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SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

DIFFERENTIAL AND INTEGRAL

CALCULUS

Third Edition

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Preface

This third edition of the well-known calculus review book by Frank Ayres, Jr., has been thoroughly revised and includes many new features. Here are some of the more significant changes:

1. Analytic geometry, knowledge of which was presupposed in the first two editions, is now treated in detail from the beginning. Chapters 1 through 5 are completely new and introduce the reader to the basic ideas and results.
2. Exponential and logarithmic functions are now treated in two places. They are first discussed briefly in Chapter 14, in the classical manner of earlier editions. Then, in Chapter 40, they are introduced and studied rigorously as is now customary in calculus courses. A thorough treatment of exponential growth and decay also is included in that chapter.
3. Terminology, notation, and standards of rigor have been brought up to date. This is especially true in connection with limits, continuity, the chain rule, and the derivative tests for extreme values.
4. Definitions of the trigonometric functions and information about the important trigonometric identities have been provided.
5. The chapter on curve tracing has been thoroughly revised, with the emphasis shifted from singular points to examples that occur more frequently in current calculus courses.

The purpose and method of the original text have nonetheless been preserved. In particular, the direct and concise exposition typical of the Schaum Outline Series has been retained. The basic aim is to offer to students a collection of carefully solved problems that are representative of those they will encounter in elementary calculus courses (generally, the first two or three semesters of a calculus sequence). Moreover, since all fundamental concepts are defined and the most important theorems are proved, this book may be used as a text for a regular calculus course, in both colleges and secondary schools.

Each chapter begins with statements of definitions, principles, and theorems. These are followed by the solved problems that form the core of the book. They give step-by-step practice in applying the principles and provide derivations of some of the theorems. In choosing these problems, we have attempted to anticipate the difficulties that normally beset the beginner. Every chapter ends with a carefully selected group of supplementary problems (with answers) whose solution is essential to the effective use of this book.

Elliott Mendelson

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Chapter 1

Absolute Value; Linear Coordinate Systems; Inequalities

THE SET OF REAL NUMBERS consists of the rational numbers (the fractions a/b, where a and b are integers) and the irrational numbers (such as $\sqrt{2} = 1.4142\ldots$ and $\pi = 3.14159\ldots$), which are not ratios of integers. Imaginary numbers, of the form $x + y\sqrt{-1}$, will not be considered. Since no confusion can result, the word *number* will always mean *real number* here.

THE ABSOLUTE VALUE $|x|$ of a number x is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \text{ is zero or a positive number} \\ -x & \text{if } x \text{ is a negative number} \end{cases}$$

For example, $|3| = |-3| = 3$ and $|0| = 0$.

In general, if x and y are any two numbers, then

$$-|x| \le x \le |x| \tag{1.1}$$

$$|-x| = |x| \qquad \text{and} \qquad |x - y| = |y - x| \tag{1.2}$$

$$|x| = |y| \text{ implies } x = \pm y \tag{1.3}$$

$$|xy| = |x| \cdot |y| \qquad \left|\frac{x}{y}\right| = \frac{|x|}{|y|} \text{ if } y \neq 0 \tag{1.4}$$

$$|x + y| \le |x| + |y| \quad \text{(Triangle inequality)} \tag{1.5}$$

A LINEAR COORDINATE SYSTEM is a graphical representation of the real numbers as the points of a straight line. To each number corresponds one and only one point, and conversely.

To set up a linear coordinate system on a given line: (1) select any point of the line as the *origin* (corresponding to 0); (2) choose a positive direction (indicated by an arrow); and (3) choose a fixed distance as a unit of measure. If x is a positive number, find the point corresponding to x by moving a distance of x units from the origin in the positive direction. If x is negative, find the point corresponding to x by moving a distance of $|x|$ units from the origin in the negative direction. (See Fig. 1-1.)

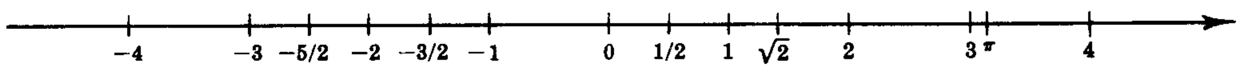


Fig. 1-1

The number assigned to a point on such a line is called the *coordinate* of that point. We often will make no distinction between a point and its coordinate. Thus, we might refer to "the point 3" rather than to "the point with coordinate 3."

If points P_1 and P_2 on the line have coordinates x_1 and x_2 (as in Fig. 1-2), then

$$|x_1 - x_2| = \overline{P_1P_2} = \text{distance between } P_1 \text{ and } P_2 \tag{1.6}$$

As a special case, if x is the coordinate of a point P, then

$$|x| = \text{distance between } P \text{ and the origin} \tag{1.7}$$

Fig. 1-2

FINITE INTERVALS. Let a and b be two points such that $a < b$. By the *open interval* (a, b) we mean the set of all points between a and b, that is, the set of all x such that $a < x < b$. By the *closed interval* $[a, b]$ we mean the set of all points between a and b or equal to a or b, that is, the set of all x such that $a \leq x \leq b$. (See Fig. 1-3.) The points a and b are called the *endpoints* of the intervals (a, b) and $[a, b]$.

Open interval (a, b): $a < x < b$ Closed interval $[a, b]$: $a \leq x \leq b$

Fig. 1-3

By a *half-open interval* we mean an open interval (a, b) together with one of its endpoints. There are two such intervals: $[a, b)$ is the set of all x such that $a \leq x < b$, and $(a, b]$ is the set of all x such that $a < x \leq b$.

For any positive number c,

$$|x| \leq c \text{ if and only if } -c \leq x \leq c \tag{1.8}$$

$$|x| < c \text{ if and only if } -c < x < c \tag{1.9}$$

See Fig. 1-4.

Fig. 1-4

INFINITE INTERVALS. Let a be any number. The set of all points x such that $a < x$ is denoted by (a, ∞); the set of all points x such that $a \leq x$ is denoted by $[a, \infty)$. Similarly, $(-\infty, b)$ denotes the set of all points x such that $x < b$, and $(-\infty, b]$ denotes the set of all x such that $x \leq b$.

INEQUALITIES such as $2x - 3 > 0$ and $5 < 3x + 10 \leq 16$ define intervals on a line, with respect to a given coordinate system.

EXAMPLE 1: Solve $2x - 3 > 0$.

$$\begin{aligned} 2x - 3 &> 0 \\ 2x &> 3 \quad \text{(Adding 3)} \\ x &> \tfrac{3}{2} \quad \text{(Dividing by 2)} \end{aligned}$$

Thus, the corresponding interval is $(\frac{3}{2}, \infty)$.

EXAMPLE 2: Solve $5 < 3x + 10 \leq 16$.

$$\begin{array}{rcll} 5 < & 3x + 10 & \leq 16 & \\ -5 < & 3x & \leq 6 & \text{(Subtracting 10)} \\ -\frac{5}{3} < & x & \leq 2 & \text{(Dividing by 3)} \end{array}$$

Thus, the corresponding interval is $(-5/3, 2]$.

EXAMPLE 3: Solve $-2x + 3 < 7$.

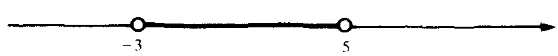
$$\begin{array}{rl} -2x + 3 < 7 & \\ -2x < 4 & \text{(Subtracting 3)} \\ x > -2 & \text{(Dividing by } -2\text{)} \end{array}$$

Note, in the last step, that division by a negative number reverses an inequality (as does multiplication by a negative number).

Solved Problems

1. Describe and diagram the following intervals, and write their interval notation: (*a*) $-3 < x < 5$; (*b*) $2 \leq x \leq 6$; (*c*) $-4 < x \leq 0$; (*d*) $x > 5$; (*e*) $x \leq 2$; (*f*) $3x - 4 \leq 8$; (*g*) $1 < 5 - 3x < 11$.

(*a*) All numbers greater than -3 and less than 5; the interval notation is $(-3, 5)$:



(*b*) All numbers equal to or greater than 2 and less than or equal to 6; $[2, 6]$:

(*c*) All numbers greater than -4 and less than or equal to 0; $(-4, 0]$:

(*d*) All numbers greater than 5; $(5, \infty)$:

(*e*) All numbers less than or equal to 2; $(-\infty, 2]$:

(*f*) $3x - 4 \leq 8$ is equivalent to $3x \leq 12$ and, therefore, to $x \leq 4$. Thus, we get $(-\infty, 4]$:

(*g*)

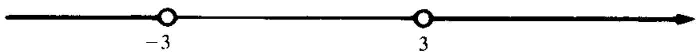
$$\begin{array}{rcll} 1 < & 5 - 3x & < 11 & \\ -4 < & -3x & < 6 & \text{(Subtracting 5)} \\ -2 < & x & < \frac{4}{3} & \text{(Dividing by } -3\text{; note the reversal of inequalities)} \end{array}$$

Thus, we obtain $(-2, \frac{4}{3})$:

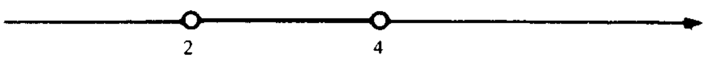
2. Describe and diagram the intervals determined by the following inequalities: (*a*) $|x| < 2$; (*b*) $|x| > 3$; (*c*) $|x - 3| < 1$; (*d*) $|x - 2| < \delta$, where $\delta > 0$; (*e*) $|x + 2| \leq 3$; (*f*) $0 < |x - 4| < \delta$, where $\delta < 0$.

(*a*) This is equivalent to $-2 < x < 2$, defining the open interval $(-2, 2)$:

(*b*) This is equivalent to $x > 3$ or $x < -3$, defining the union of the infinite intervals $(3, \infty)$ and $(-\infty, -3)$.



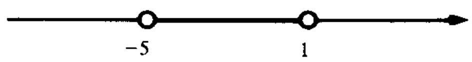
(*c*) This is equivalent to saying that the distance between x and 3 is less than 1, or that $2 < x < 4$, which defines the open interval $(2, 4)$:



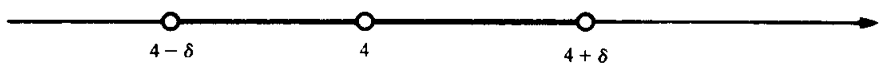
We can also note that $|x - 3| < 1$ is equivalent to $-1 < x - 3 < 1$. Adding 3, we obtain $2 < x < 4$.

(*d*) This is equivalent to saying that the distance between x and 2 is less than δ, or that $2 - \delta < x < 2 + \delta$, which defines the open interval $(2 - \delta, 2 + \delta)$. This interval is called the *δ-neighborhood* of 2:

(*e*) $|x + 2| < 3$ is equivalent to $-3 < x + 2 < 3$. Subtracting 2, we obtain $-5 < x < 1$, which defines the open interval $(-5, 1)$:

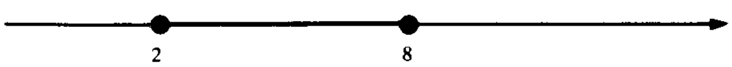


(*f*) The inequality $|x - 4| < \delta$ determines the interval $4 - \delta < x < 4 + \delta$. The additional condition $0 < |x - 4|$ tells us that $x \neq 4$. Thus, we get the union of the two intervals $(4 - \delta, 4)$ and $(4, 4 + \delta)$. The result is called the *deleted δ-neighborhood* of 4:

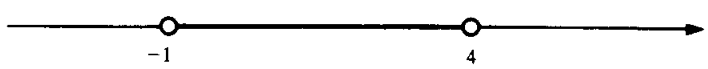


3. Describe and diagram the intervals determined by the following inequalities: (*a*) $|5 - x| \leq 3$; (*b*) $|2x - 3| < 5$; (*c*) $|1 - 4x| < \frac{1}{2}$.

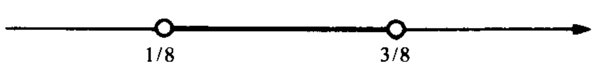
(*a*) Since $|5 - x| = |x - 5|$, we have $|x - 5| \leq 3$, which is equivalent to $-3 \leq x - 5 \leq 3$. Adding 5, we get $2 \leq x \leq 8$, which defines the open interval $(2, 8)$:



(*b*) $|2x - 3| < 5$ is equivalent to $-5 < 2x - 3 < 5$. Adding 3, we have $-2 < 2x < 8$; then dividing by 2 yields $-1 < x < 4$, which defines the open interval $(-1, 4)$:



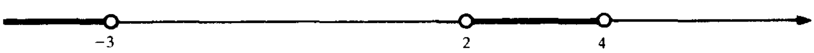
(*c*) Since $|1 - 4x| = |4x - 1|$, we have $|4x - 1| < \frac{1}{2}$, which is equivalent to $-\frac{1}{2} < 4x - 1 < \frac{1}{2}$. Adding 1, we get $\frac{1}{2} < 4x < \frac{3}{2}$. Dividing by 4, we obtain $\frac{1}{8} < x < \frac{3}{8}$, which defines the interval $(\frac{1}{8}, \frac{3}{8})$:



4. Solve the inequalities (*a*) $18x - 3x^2 > 0$, (*b*) $(x+3)(x-2)(x-4) < 0$, and (*c*) $(x+1)^2(x-3) > 0$, and diagram the solutions.

(*a*) Set $18x - 3x^2 = 3x(6-x) = 0$, obtaining $x = 0$ and $x = 6$. We need to determine the sign of $18x - 3x^2$ on each of the intervals $x < 0$, $0 < x < 6$, and $x > 6$, to determine where $18x - 3x^2 > 0$. We note that it is negative when $x < 0$, and that it changes sign when we pass through 0 and 6. Hence, it is positive when and only when $0 < x < 6$:

(*b*) The crucial points are $x = -3$, $x = 2$, and $x = 4$. Note that $(x+3)(x-2)(x-4)$ is negative for $x < -3$ (since each of the factors is negative) and that it changes sign when we pass through each of the crucial points. Hence, it is negative for $x < -3$ and for $2 < x < 4$:



(*c*) Note that $(x+1)^2$ is always positive (except at $x = -1$, where it is 0). Hence $(x+1)^2(x-3) > 0$ when and only when $x - 3 > 0$, that is, for $x > 3$:

5. Solve $|3x - 7| = 8$.

In general, when $c \geq 0$, $|u| = c$ if and only if $u = c$ or $u = -c$. Thus, we need to solve $3x - 7 = 8$ and $3x - 7 = -8$, from which we get $x = 5$ or $x = -\frac{1}{3}$.

6. Solve $\dfrac{2x+1}{x+3} > 3$.

Case 1: $x + 3 > 0$. Multiply by $x + 3$ to obtain $2x + 1 > 3x + 9$, which reduces to $-8 > x$. However, since $x + 3 > 0$, it must be that $x > -3$. Thus, this case yields no solutions.

Case 2: $x + 3 < 0$. Multiply by $x + 3$ to obtain $2x + 1 < 3x + 9$. (Note that the inequality is reversed, since we multiplied by a negative number.) This yields $-8 < x$. Since $x + 3 < 0$, we have $x < -3$. Thus, the only solutions are $-8 < x < -3$.

7. Solve $\left|\dfrac{2}{x} - 3\right| < 5$.

The given inequality is equivalent to $-5 < \dfrac{2}{x} - 3 < 5$. Add 3 to obtain $-2 < 2/x < 8$, and divide by 2 to get $-1 < 1/x < 4$.

Case 1: $x > 0$. Multiply by x to get $-x < 1 < 4x$. Then $x > \frac{1}{4}$ and $x > -1$; these two inequalities are equivalent to the single inequality $x > \frac{1}{4}$.

Case 2: $x < 0$. Multiply by x to obtain $-x > 1 > 4x$. (Note that the inequalities have been reversed, since we multiplied by the negative number x.) Then $x < \frac{1}{4}$ and $x < -1$. These two inequalities are equivalent to $x < -1$.

Thus, the solutions are $x > \frac{1}{4}$ or $x < -1$, the union of the two infinite intervals $(\frac{1}{4}, \infty)$ and $(-\infty, -1)$.

8. Solve $|2x - 5| \geq 3$.

Let us first solve the negation $|2x - 5| < 3$. The latter is equivalent to $-3 < 2x - 5 < 3$. Add 5 to obtain $2 < 2x < 8$, and divide by 2 to obtain $1 < x < 4$. Since this is the solution of the negation, the original inequality has the solution $x \leq 1$ or $x \geq 4$.

9. Prove the triangle inequality, $|x + y| \leq |x| + |y|$.

Add the inequalities $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$ to obtain

$$-(|x|+|y|) \leq x+y \leq |x|+|y|$$

Then, by (*1.8*), $|x+y| \leq |x|+|y|$.

Supplementary Problems

10. Describe and diagram the set determined by each of the following conditions:

(*a*) $-5 < x < 0$ (*b*) $x \leq 0$ (*c*) $-2 \leq x < 3$ (*d*) $x \geq 1$
(*e*) $|x| < 3$ (*f*) $|x| \geq 5$ (*g*) $|x-2| < \frac{1}{2}$ (*h*) $|x+3| > 1$
(*i*) $0 < |x-2| < 1$ (*j*) $0 < |x+3| < \frac{1}{4}$ (*k*) $|x-2| \geq 1$.

Ans. (*e*) $-3 < x < 3$; (*f*) $x \geq 5$ or $x \leq -5$; (*g*) $\frac{3}{2} < x < \frac{5}{2}$; (*h*) $x > -2$ or $x < -4$; (*i*) $x \neq 2$ and $1 < x < 3$; (*j*) $-\frac{13}{4} < x < -\frac{11}{4}$; (*k*) $x \geq 3$ or $x \leq 1$

11. Describe and diagram the set determined by each of the following conditions:

(*a*) $|3x-7| < 2$ (*b*) $|4x-1| \geq 1$ (*c*) $\left|\frac{x}{3}-2\right| \leq 4$
(*d*) $\left|\frac{3}{x}-2\right| \leq 4$ (*e*) $\left|2+\frac{1}{x}\right| > 1$ (*f*) $\left|\frac{4}{x}\right| < 3$

Ans. (*a*) $\frac{5}{3} < x < 3$; (*b*) $x \geq \frac{1}{2}$ or $x \leq 0$; (*c*) $-6 \leq x \leq 18$; (*d*) $x \leq -\frac{3}{2}$ or $x \geq \frac{1}{2}$; (*e*) $x > 0$ or $x < -1$ or $-\frac{1}{3} < x < 0$; (*f*) $x > \frac{4}{3}$ or $x < -\frac{4}{3}$

12. Describe and diagram the set determined by each of the following conditions:

(*a*) $x(x-5) < 0$ (*b*) $(x-2)(x-6) > 0$ (*c*) $(x+1)(x-2) < 0$
(*d*) $x(x-2)(x+3) > 0$ (*e*) $(x+2)(x+3)(x+4) < 0$ (*f*) $(x-1)(x+1)(x-2)(x+3) > 0$
(*g*) $(x-1)^2(x+4) > 0$ (*h*) $(x-3)(x+5)(x-4)^2 < 0$ (*i*) $(x-2)^3 > 0$
(*j*) $(x+1)^3 < 0$ (*k*) $(x-2)^3(x+1) < 0$ (*l*) $(x-1)^3(x+1)^4 < 0$
(*m*) $(3x-1)(2x+3) > 0$ (*n*) $(x-4)(2x-3) < 0$

Ans. (*a*) $0 < x < 5$; (*b*) $x > 6$ or $x < 2$; (*c*) $-1 < x < 2$; (*d*) $x > 2$ or $-3 < x < 0$; (*e*) $-3 < x < -2$ or $x < -4$; (*f*) $x > 2$ or $-1 < x < 1$ or $x < -3$; (*g*) $x > -4$ and $x \neq 1$; (*h*) $-5 < x < 3$; (*i*) $x > 2$; (*j*) $x < -1$; (*k*) $-1 < x < 2$; (*l*) $x < 1$ and $x \neq -1$; (*m*) $x > \frac{1}{3}$ or $x < -\frac{3}{2}$; (*n*) $\frac{3}{2} < x < 4$

13. Describe and diagram the set determined by each of the following conditions:

(*a*) $x^2 < 4$ (*b*) $x^2 \geq 9$ (*c*) $(x-2)^2 \leq 16$ (*d*) $(2x+1)^2 > 1$
(*e*) $x^2+3x-4 > 0$ (*f*) $x^2+6x+8 \leq 0$ (*g*) $x^2 < 5x+14$ (*h*) $2x^2 > x+6$
(*i*) $6x^2+13x < 5$ (*j*) $x^3+3x^2 > 10x$

Ans. (*a*) $-2 < x < 2$; (*b*) $x \geq 3$ or $x \leq -3$; (*c*) $-2 \leq x \leq 6$; (*d*) $x > 0$ or $x < -1$; (*e*) $x > 1$ or $x < -4$; (*f*) $-4 \leq x \leq -2$; (*g*) $-2 < x < 7$; (*h*) $x > 2$ or $x < -\frac{3}{2}$; (*i*) $-\frac{5}{2} < x < \frac{1}{3}$; (*j*) $-5 < x < 0$ or $x > 2$

14. Solve: (*a*) $-4 < 2-x < 7$ (*b*) $\frac{2x-1}{x} < 3$ (*c*) $\frac{x}{x+2} < 1$

(*d*) $\frac{3x-1}{2x+3} > 3$ (*e*) $\left|\frac{2x-1}{x}\right| > 2$ (*f*) $\left|\frac{x}{x+2}\right| \leq 2$

Ans. (*a*) $-5<x<6$; (*b*) $x>0$ or $x<-1$; (*c*) $x>-2$; (*d*) $-\frac{10}{3}<x<-\frac{3}{2}$; (*e*) $x<0$ or $0<x<\frac{1}{4}$; (*f*) $x\leq-4$ or $x\geq-1$

15. Solve: (*a*) $|4x-5|=3$ (*b*) $|x+6|=2$ (*c*) $|3x-4|=|2x+1|$ (*d*) $|x+1|=|x+2|$ (*e*) $|x+1|=3x-1$ (*f*) $|x+1|<|3x-1|$ (*g*) $|3x-4|\geq|2x+1|$

Ans. (*a*) $x=2$ or $x=\frac{1}{2}$; (*b*) $x=-4$ or $x=-8$; (*c*) $x=5$ or $x=\frac{3}{5}$; (*d*) $x=-\frac{3}{2}$; (*e*) $x=1$; (*f*) $x>1$ or $x<0$; (*g*) $x>5$ or $x<\frac{3}{5}$

16. Prove: (*a*) $|xy|=|x|\cdot|y|$ (*b*) $\left|\frac{x}{y}\right|=\frac{|x|}{|y|}$ if $y\neq0$ (*c*) $|x^2|=|x|^2$ (*d*) $|x-y|\leq|x|+|y|$ (*e*) $|x-y|\geq||x|-|y||$
(*Hint*: In (*e*), prove that $|x-y|\geq|x|-|y|$ and $|x-y|\geq|y|-|x|$.)

Chapter 2

The Rectangular Coordinate System

COORDINATE AXES. In any plane $\mathscr{P}$, choose a pair of perpendicular lines. Let one of the lines be horizontal. Then the other line must be vertical. The horizontal line is called the *x axis*, and the vertical line the *y axis*. (See Fig. 2-1.)

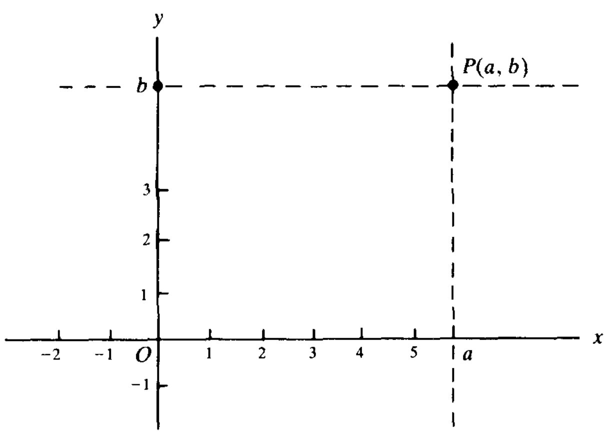


Fig. 2-1

Now choose linear coordinate systems on the x axis and the y axis satisfying the following conditions: The origin for each coordinate system is the point O at which the axes intersect. The x axis is directed from left to right, and the y axis from bottom to top. The part of the x axis with positive coordinates is called the *positive x axis*, and the part of the y axis with positive coordinates is called the *positive y axis*.

We shall establish a correspondence between the points of the plane $\mathscr{P}$ and pairs of real numbers.

COORDINATES. Consider any point P of the plane (Fig. 2-1). The vertical line through P intersects the x axis at a unique point; let a be the coordinate of this point on the x axis. The number a is called the *x coordinate* of P (or the *abscissa* of P). The horizontal line through P intersects the y axis at a unique point; let b be the coordinate of this point on the y axis. The number b is called the *y coordinate* of P (or the *ordinate* of P). In this way, every point P has a unique pair (a, b) of real numbers associated with it. Conversely, every pair (a, b) of real numbers is associated with a unique point in the plane.

The coordinates of several points are shown in Fig. 2-2. For the sake of simplicity, we have limited them to integers.

EXAMPLE 1: In the coordinate system of Fig. 2-3, to find the point having coordinates $(2, 3)$, start at the origin, move two units to the *right*, and then three units *upward*.

To find the point with coordinates $(-4, 2)$, start at the origin, move four units to the *left*, and then two units *upward*.

To find the point with coordinates $(-3, -1)$, start at the origin, move three units to the *left*, and then one unit *downward*.

The order of these moves is not important. Hence, for example, the point $(2, 3)$ can also be reached by starting at the origin, moving three units *upward*, and then two units to the *right*.

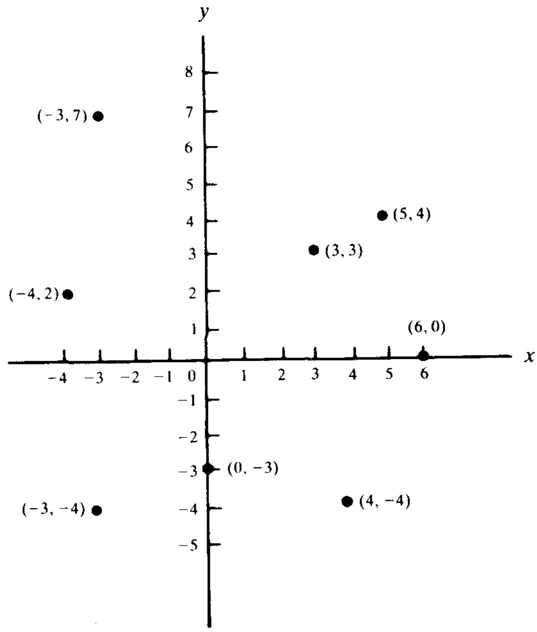


Fig. 2-2

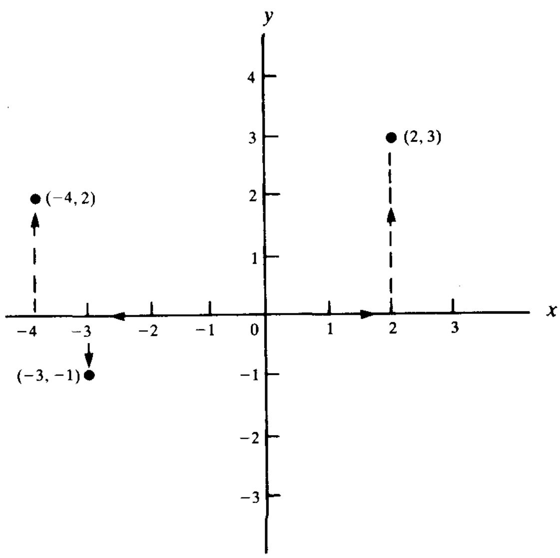


Fig. 2-3

QUADRANTS. Assume that a coordinate system has been established in the plane $\mathcal{P}$. Then the whole plane $\mathcal{P}$, with the exception of the coordinate axes, can be divided into four equal parts, called *quadrants*. All points with both coordinates positive form the first quadrant, called *quadrant I*, in the upper right-hand corner. (See Fig. 2-4.) *Quadrant II* consists of all points with negative x coordinate and positive y coordinate. *Quadrants III* and *IV* are also shown in Fig. 2-4.

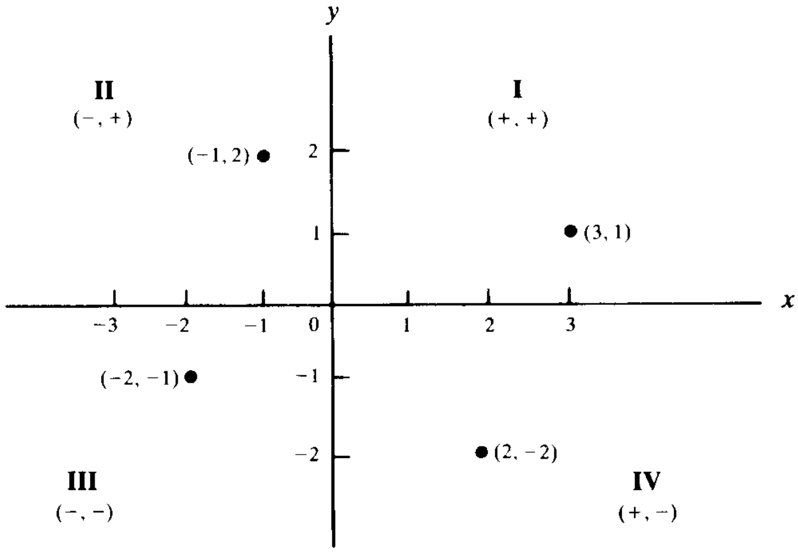


Fig. 2-4

The points on the x axis have coordinates of the form $(a, 0)$. The y axis consists of the points with coordinates of the form $(0, b)$.

Given a coordinate system, it is customary to refer to the point with coordinates (a, b) as "the point (a, b)." For example, one might say, "The point $(0, 1)$ lies on the y axis."

DISTANCE FORMULA. The distance $\overline{P_1P_2}$ between points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) is

$$\overline{P_1P_2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{2.1}$$

EXAMPLE 2: (*a*) The distance between $(2, 5)$ and $(7, 17)$ is

$$\sqrt{(2-7)^2 + (5-17)^2} = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

(*b*) The distance between $(1, 4)$ and $(5, 2)$

$$\sqrt{(1-5)^2 + (4-2)^2} = \sqrt{(-4)^2 + (2)^2} = \sqrt{16+4} = \sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$$

MIDPOINT FORMULAS. The point $M(x, y)$ that is the midpoint of the segment connecting the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ has coordinates

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2} \tag{2.2}$$

The coordinates of the midpoint are the averages of the coordinates of the endpoints.

EXAMPLE 3: (*a*) The midpoint of the segment connecting $(2, 9)$ and $(4, 3)$ is $\left(\frac{2+4}{2}, \frac{9+3}{2}\right) = (3, 6)$.

(*b*) The point halfway between $(-5, 1)$ and $(1, 4)$ is $\left(\frac{-5+1}{2}, \frac{1+4}{2}\right) = \left(-2, \frac{5}{2}\right)$.

PROOFS OF GEOMETRIC THEOREMS can often be given more easily by use of coordinates than by deduction from axioms and previously derived theorems. Proofs by means of coordinates are called *analytic*, in contrast to the so-called *synthetic* proofs from axioms.

EXAMPLE 4: Let us prove analytically that the segment joining the midpoints of two sides of a triangle is one-half the length of the third side. Construct a coordinate system so that the third side AB lies on the positive x axis, A is the origin, and the third vertex C lies above the x axis, as in Fig. 2-5.

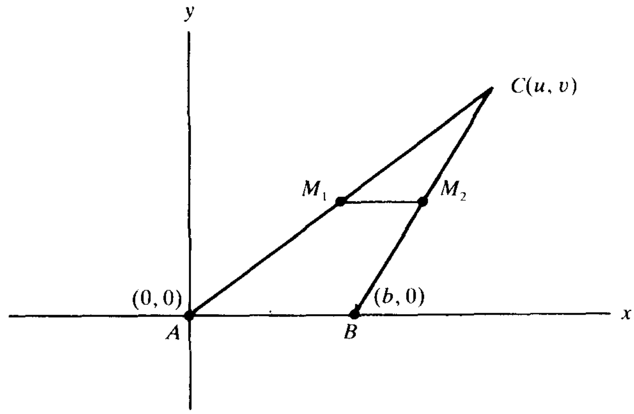


Fig. 2-5

Let b be the x coordinate of B. (In other words, let $b = \overline{AB}$.) Let C have coordinates (u, v). Let M_1 and M_2 be the midpoints of sides AC and BC, respectively. By the midpoint formulas *(2.2)*, the coordinates of M_1 are $\left(\frac{u}{2}, \frac{v}{2}\right)$, and the coordinates of M_2 are $\left(\frac{u+b}{2}, \frac{v}{2}\right)$. By the distance formula *(2.1)*,

$$\overline{M_1M_2} = \sqrt{\left(\frac{u}{2} - \frac{u+b}{2}\right)^2 + \left(\frac{v}{2} - \frac{v}{2}\right)^2} = \sqrt{\left(\frac{b}{2}\right)^2} = \frac{b}{2}$$

which is half the length of side AB.

Solved Problems

1. Derive the distance formula *(2.1)*.

Given points P_1 and P_2 in Fig. 2-6, let Q be the point at which the vertical line through P_2 intersects the horizontal line through P_1. The x coordinate of Q is x_2, the same as that of P_2. The y coordinate of Q is y_1, the same as that of P_1.

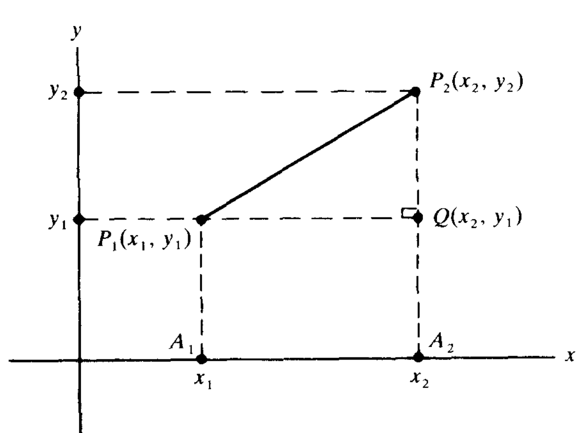


Fig. 2-6

By the Pythagorean theorem,

$$(\overline{P_1P_2})^2 = (\overline{P_1Q})^2 + (\overline{P_2Q})^2 \tag{1}$$

If A_1 and A_2 are the projections of P_1 and P_2 on the x axis, then the segments P_1Q and A_1A_2 are opposite sides of a rectangle. Hence, $\overline{P_1Q} = \overline{A_1A_2}$. But $\overline{A_1A_2} = |x_1 - x_2|$ by (*1.6*). Therefore, $\overline{P_1Q} = |x_1 - x_2|$. By similar reasoning, $\overline{P_2Q} = |y_1 - y_2|$. Hence, by (*1*),

$$\overline{P_1P_2}^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Taking square roots yields the distance formula (*2.1*).

2. Show that the distance between a point $P(x, y)$ and the origin is $\sqrt{x^2 + y^2}$.

Since the origin has coordinates $(0, 0)$, the distance formula yields $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.

3. Prove the midpoint formulas (*2.2*).

We wish to find the coordinates (x, y) of the midpoint M of the segment P_1P_2 in Fig. 2-7. Let A, B, and C be the perpendicular projections of P_1, M, and P_2 on the x axis.

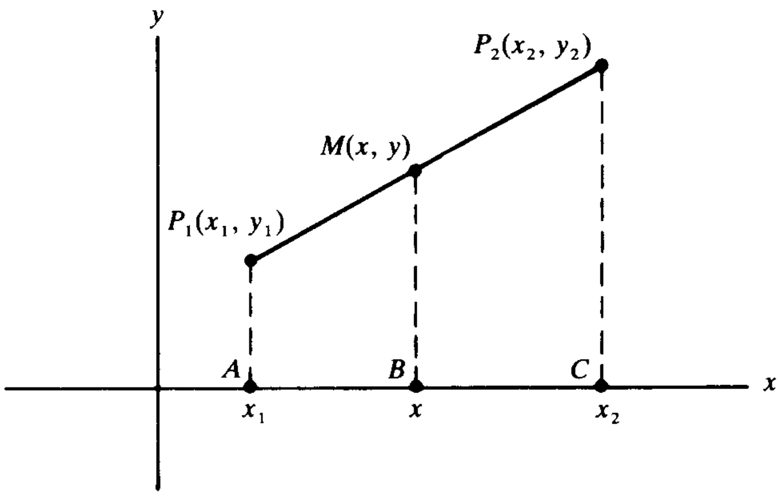


Fig. 2-7

The x coordinates of A, B, and C are x_1, x, and x_2, respectively. Since the lines P_1A, MB, and P_2C are parallel, the ratios $\overline{P_1M}/\overline{MP_2}$ and $\overline{AB}/\overline{BC}$ are equal. (In general, if two lines are intersected by three parallel lines, the ratios of corresponding segments are equal.) But, $\overline{P_1M} = \overline{MP_2}$. Hence, $\overline{AB} = \overline{BC}$. Since $\overline{AB} = x - x_1$ and $\overline{BC} = x_2 - x$, we obtain $x - x_1 = x_2 - x$, and therefore $2x = x_1 + x_2$. Dividing by 2, we get $x = (x_1 + x_2)/2$. (We obtain the same result when P_2 is to the left of P_1. In that case, $\overline{AB} = x_1 - x$ and $\overline{BC} = x - x_2$.) A similar argument shows that $y = (y_1 + y_2)/2$.

4. Is the triangle with vertices $A(1, 5)$, $B(4, 2)$, and $C(5, 6)$ isosceles?

$$\overline{AB} = \sqrt{(1-4)^2 + (5-2)^2} = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18}$$
$$\overline{AC} = \sqrt{(1-5)^2 + (5-6)^2} = \sqrt{(-4)^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}$$
$$\overline{BC} = \sqrt{(4-5)^2 + (2-6)^2} = \sqrt{(-1)^2 + (-4)^2} = \sqrt{1+16} = \sqrt{17}$$

Since $\overline{AC} = \overline{BC}$, the triangle is isosceles.

5. Is the triangle with vertices $A(-5, 6)$, $B(2, 3)$, and $C(5, 10)$ a right triangle?

$$\overline{AB} = \sqrt{(-5-2)^2 + (6-3)^2} = \sqrt{(-7)^2 + (3)^2} = \sqrt{49+9} = \sqrt{58}$$
$$\overline{AC} = \sqrt{(-5-5)^2 + (6-10)^2} = \sqrt{(-10)^2 + (-4)^2} = \sqrt{100+16} = \sqrt{116}$$
$$\overline{BC} = \sqrt{(2-5)^2 + (3-10)^2} = \sqrt{(-3)^2 + (-7)^2} = \sqrt{9+49} = \sqrt{58}$$

Since $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, the converse of the Pythagorean theorem tells us that $\triangle ABC$ is a right triangle, with right angle at B; in fact, since $\overline{AB} = \overline{BC}$, $\triangle ABC$ is an isosceles right triangle.

6. Prove analytically that, if the medians to two sides of a triangle are equal, then those sides are equal. (Recall that a *median* of a triangle is a line segment joining a vertex to the midpoint of the opposite side.)

In $\triangle ABC$, let M_1 and M_2 be the midpoints of sides AC and BC, respectively. Construct a coordinate system so that A is the origin, B lies on the positive x axis, and C lies above the x axis (see Fig. 2-8). Assume that $\overline{AM_2} = \overline{BM_1}$. We must prove that $\overline{AC} = \overline{BC}$. Let b be the x coordinate of B, and let C have coordinates (u, v). Then, by the midpoint formulas, M_1 has coordinates $\left(\frac{u}{2}, \frac{v}{2}\right)$, and M_2 has coordinates $\left(\frac{u+b}{2}, \frac{v}{2}\right)$. Hence,

$$\overline{AM_2} = \sqrt{\left(\frac{u+b}{2}\right)^2 + \left(\frac{v}{2}\right)^2} \quad \text{and} \quad \overline{BM_1} = \sqrt{\left(\frac{u}{2} - b\right)^2 + \left(\frac{v}{2}\right)^2}$$

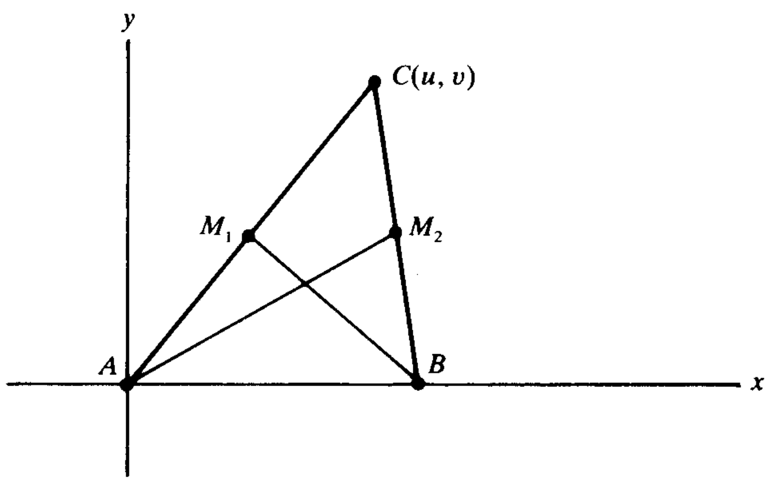


Fig. 2-8

Since $\overline{AM_2} = \overline{BM_1}$,

$$\left(\frac{u+b}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = \left(\frac{u}{2} - b\right)^2 + \left(\frac{v}{2}\right)^2 = \left(\frac{u-2b}{2}\right)^2 + \left(\frac{v}{2}\right)^2$$

Hence, $\frac{(u+b)^2}{4} + \frac{v^2}{4} = \frac{(u-2b)^2}{4} + \frac{v^2}{4}$ and, therefore, $(u+b)^2 = (u-2b)^2$. So, $u + b = \pm(u-2b)$. If $u + b = u - 2b$, then $b = -2b$, and therefore, $b = 0$, which is impossible, since $A \neq B$. Hence, $u + b = -(u-2b) = -u + 2b$, whence $2u = b$. Now $\overline{BC} = \sqrt{(u-b)^2 + v^2} = \sqrt{(u-2u)^2 + v^2} = \sqrt{(-u)^2 + v^2} = \sqrt{u^2 + v^2}$, and $\overline{AC} = \sqrt{u^2 + v^2}$. Thus, $\overline{AC} = \overline{BC}$.

7. Find the coordinates (x, y) of the point Q on the line segment joining $P_1(1, 2)$ and $P_2(6, 7)$, such that Q divides the segment in the ratio 2:3, that is, such that $\overline{P_1Q}/\overline{QP_2} = 2/3$.

Let the projections of P_1, Q, and P_2 on the x axis be A_1, Q', and A_2, with x coordinates 1, x, and 6, respectively (see Fig. 2-9). Now $\overline{A_1Q'}/\overline{Q'A_2} = \overline{P_1Q}/\overline{QP_2} = 2/3$. (When two lines are cut by three parallel lines, corresponding segments are in proportion.) But $\overline{A_1Q'} = x - 1$, and $\overline{Q'A_2} = 6 - x$. So

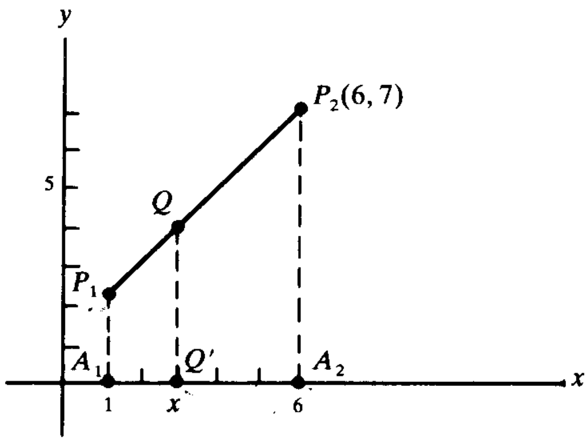


Fig. 2-9

$\dfrac{x-1}{6-x} = \dfrac{2}{3}$, and cross-multiplying yields $3x - 3 = 12 - 2x$. Hence $5x = 15$, whence $x = 3$. By similar reasoning, $\dfrac{y-2}{7-y} = \dfrac{2}{3}$, from which it follows that $y = 4$.

Supplementary Problems

8. In Fig. 2-10, find the coordinates of points A, B, C, D, E, and F.

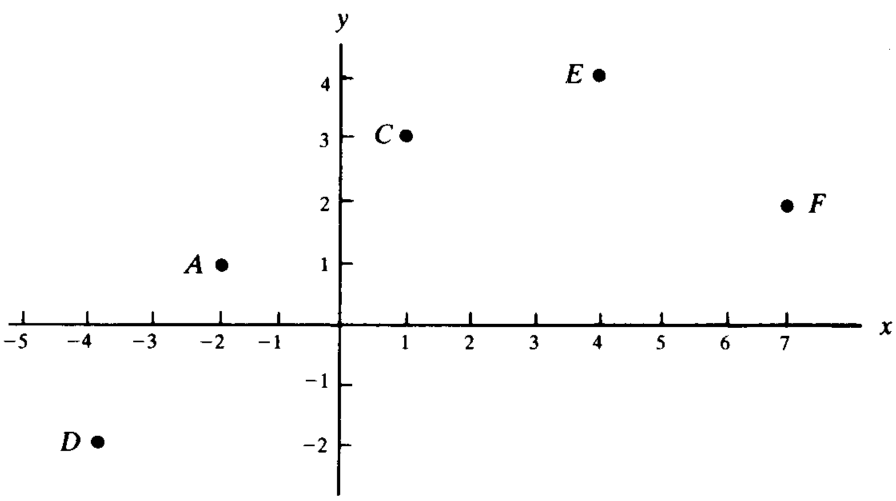


Fig. 2-10

Ans. $A = (-2, 1)$; $B = (0, -1)$; $C = (1, 3)$; $D = (-4, -2)$; $E = (4, 4)$; $F = (7, 2)$.

9. Draw a coordinate system and show the points having the following coordinates: $(2, -3)$, $(3, 3)$, $(-1, 1)$, $(2, -2)$, $(0, 3)$, $(3, 0)$, $(-2, 3)$.

10. Find the distances between the following pairs of points:
(*a*) $(3, 4)$ and $(3, 6)$ (*b*) $(2, 5)$ and $(2, -2)$ (*c*) $(3, 1)$ and $(2, 1)$
(*d*) $(2, 3)$ and $(5, 7)$ (*e*) $(-2, 4)$ and $(3, 0)$ (*f*) $(-2, \frac{1}{2})$ and $(4, -1)$

Ans. (*a*) 2; (*b*) 7; (*c*) 1; (*d*) 5; (*e*) $\sqrt{41}$; (*f*) $\frac{3}{2}\sqrt{17}$

11. Draw the triangle with vertices $A(2, 5)$, $B(2, -5)$, and $C(-3, 5)$, and find its area.

Ans. area $= 25$

12. If $(2, 2)$, $(2, -4)$, and $(5, 2)$ are three vertices of a rectangle, find the fourth vertex.

Ans. $(5, -4)$

13. If the points $(2, 4)$ and $(-1, 3)$ are opposite vertices of a rectangle whose sides are parallel to the coordinate axes (that is, the x and y axes), find the other two vertices.

Ans. $(-1, 4)$ and $(2, 3)$

14. Determine whether the following triples of points are the vertices of an isosceles triangle: (*a*) $(4, 3)$, $(1, 4)$, $(3, 10)$; (*b*) $(-1, 1)$, $(3, 3)$, $(1, -1)$; (*c*) $(2, 4)$, $(5, 2)$, $(6, 5)$.

Ans. (*a*) no; (*b*) yes; (*c*) no

15. Determine whether the following triples of points are the vertices of a right triangle. For those that are, find the area of the right triangle: (*a*) $(10, 6)$, $(3, 3)$, $(6, -4)$; (*b*) $(3, 1)$, $(1, -2)$, $(-3, -1)$; (*c*) $(5, -2)$, $(0, 3)$, $(2, 4)$.

Ans. (*a*) yes, area $= 29$; (*b*) no; (*c*) yes, area $= \frac{15}{2}$

16. Find the perimeter of the triangle with vertices $A(4, 9)$, $B(-3, 2)$, and $C(8, -5)$.

Ans. $7\sqrt{2} + \sqrt{170} + 2\sqrt{53}$

17. Find the value or values of y for which $(6, y)$ is equidistant from $(4, 2)$ and $(9, 7)$.

Ans. 5

18. Find the midpoints of the line segments with the following endpoints: (*a*) $(2, -3)$ and $(7, 4)$; (*b*) $(\frac{5}{3}, 2)$ and $(4, 1)$; (*c*) $(\sqrt{3}, 0)$ and $(1, 4)$.

Ans. (*a*) $\left(\frac{9}{2}, \frac{1}{2}\right)$; (*b*) $\left(\frac{17}{6}, \frac{3}{2}\right)$; (*c*) $\left(\frac{1+\sqrt{3}}{2}, 2\right)$

19. Find the point (x, y) such that $(2, 4)$ is the midpoint of the line segment connecting (x, y) and $(1, 5)$.

Ans. $(3, 3)$

20. Determine the point that is equidistant from the points $A(-1, 7)$, $B(6, 6)$, and $C(5, -1)$.

Ans. $(\frac{52}{25}, \frac{153}{50})$

21. Prove analytically that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

22. Show analytically that the sum of the squares of the distances of any point P from two opposite vertices of a rectangle is equal to the sum of the squares of its distances from the other two vertices.

23. Prove analytically that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

24. Prove analytically that the sum of the squares of the medians of a triangle is equal to three-fourths the sum of the squares of the sides.

25. Prove analytically that the line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

26. Prove that the coordinates (x, y) of the point Q that divides the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ in the ratio $r_1 : r_2$ are determined by the formulas $x = \dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}$ and $y = \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2}$. (*Hint*: Use the reasoning of Problem 7.)

27. Find the coordinates of the point Q on the segment P_1P_2 such that $\overline{P_1Q}/\overline{QP_2} = 2/7$, if (*a*) $P_1 = (0, 0)$, $P_2 = (7, 9)$; (*b*) $P_1 = (-1, 0)$, $P_2 = (0, 7)$; (*c*) $P_1 = (-7, -2)$, $P_2 = (2, 7)$; (*d*) $P_1 = (1, 3)$, $P_2 = (4, 2)$.

Ans. (*a*) $(\frac{14}{9}, 2)$; (*b*) $(-\frac{7}{9}, \frac{14}{9})$; (*c*) $(-5, \frac{28}{9})$; (*d*) $(\frac{13}{9}, \frac{32}{9})$

Chapter 3

Lines

THE STEEPNESS OF A LINE is measured by a number called the *slope* of the line. Let $\mathscr{L}$ be any line, and let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points of $\mathscr{L}$. The slope of $\mathscr{L}$ is defined to be the number $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. The slope is the ratio of a change in the y coordinate to the corresponding change in the x coordinate. (See Fig. 3-1.)

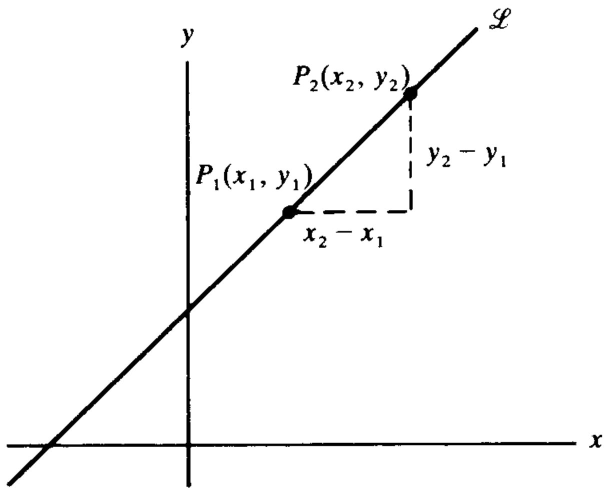


Fig. 3-1

For the definition of the slope to make sense, it is necessary to check that the number m is independent of the choice of the points P_1 and P_2. If we choose another pair $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$, the same value of m must result. In Fig. 3-2, triangle P_3P_4T is similar to triangle P_1P_2Q. Hence,

$$\frac{\overline{QP_2}}{\overline{P_1Q}} = \frac{\overline{TP_4}}{\overline{P_3T}} \quad \text{or} \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

Therefore, P_1 and P_2 determine the same slope as P_3 and P_4.

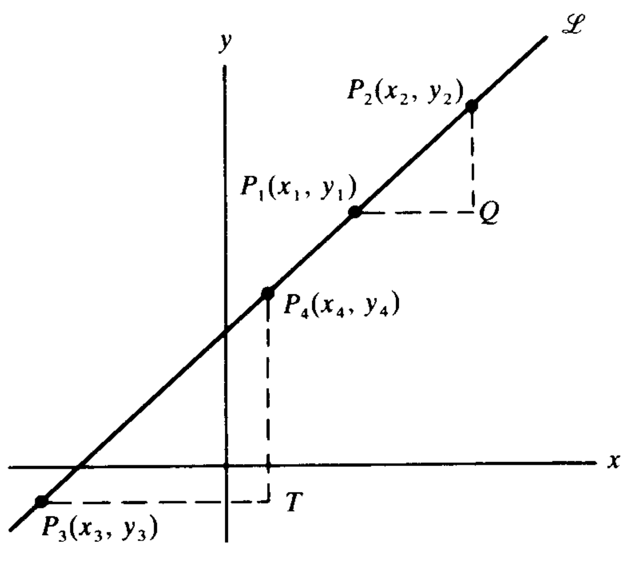


Fig. 3-2

EXAMPLE 1: The slope of the line joining the points $(1, 2)$ and $(4, 6)$ in Fig. 3-3 is $\frac{6-2}{4-1} = \frac{4}{3}$. Hence, as a point on the line moves 3 units to the right, it moves 4 units upward. Moreover, the slope is not affected by the order in which the points are given: $\frac{2-6}{1-4} = \frac{-4}{-3} = \frac{4}{3}$. In general, $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

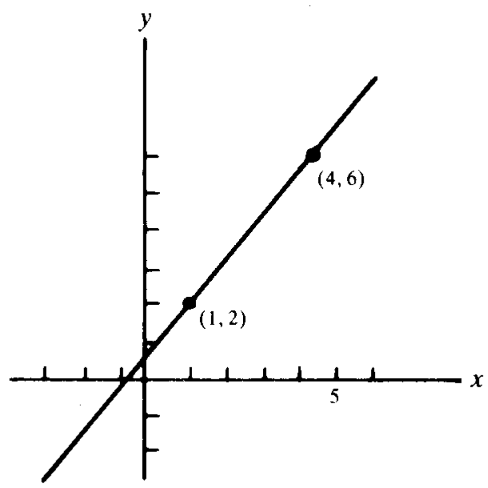


Fig. 3-3

THE SIGN OF THE SLOPE has significance. Consider, for example, a line $\mathscr{L}$ that moves upward as it moves to the right, as in Fig. 3-4(*a*). Since $y_2 > y_1$ and $x_2 > x_1$, we have $m = \frac{y_2 - y_1}{x_2 - x_1} > 0$. *The slope of* $\mathscr{L}$ *is positive.*

Now consider a line $\mathscr{L}$ that moves downward as it moves to the right, as in Fig. 3-4(*b*). Here $y_2 < y_1$ while $x_2 > x_1$; hence, $m = \frac{y_2 - y_1}{x_2 - x_1} < 0$. *The slope of* $\mathscr{L}$ *is negative.*

Now let the line $\mathscr{L}$ be horizontal, as in Fig. 3-4(*c*). Here $y_1 = y_2$, so that $y_2 - y_1 = 0$. In addition, $x_2 - x_1 \neq 0$. Hence, $m = \frac{0}{x_2 - x_1} = 0$. *The slope of* $\mathscr{L}$ *is zero.*

Line $\mathscr{L}$ is vertical in Fig. 3-4(*d*), where we see that $y_2 - y_1 > 0$ while $x_2 - x_1 = 0$. Thus, the expression $\frac{y_2 - y_1}{x_2 - x_1}$ is undefined. *The slope is not defined for a vertical line* $\mathscr{L}$. (Sometimes we describe this situation by saying that the slope of $\mathscr{L}$ is "infinite.")

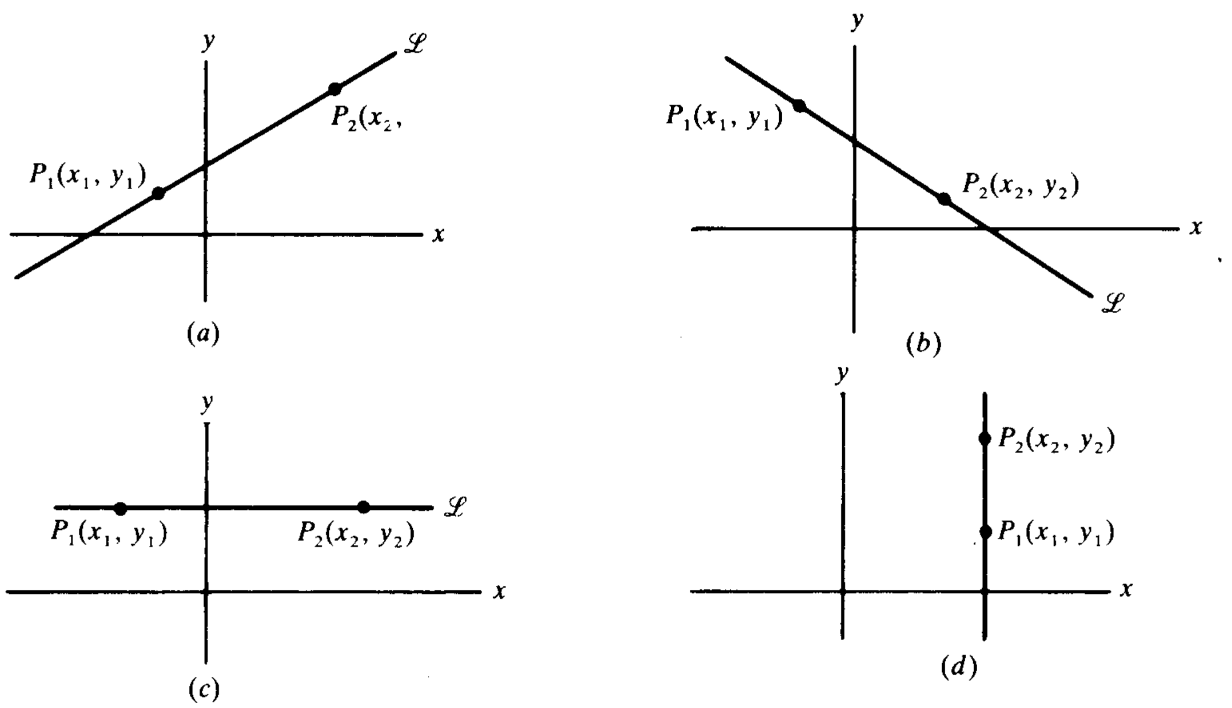


Fig. 3-4

SLOPE AND STEEPNESS. Consider any line $\mathscr{L}$ with positive slope, passing through a point $P_1(x_1, y_1)$; such a line is shown in Fig. 3-5. Choose the point $P_2(x_2, y_2)$ on $\mathscr{L}$ such that $x_2 - x_1 = 1$. Then the slope m of $\mathscr{L}$ is equal to the distance $\overline{AP_2}$. As the steepness of the line increases, $\overline{AP_2}$ increases without limit, as shown in Fig. 3-6(a). Thus, the slope of $\mathscr{L}$ increases without bound from 0 (when $\mathscr{L}$ is horizontal) to $+\infty$ (when the line is vertical). By a similar argument, using Fig. 3-6(b), we can show that as a negatively sloped line becomes steeper, the slope steadily decreases from 0 (when the line is horizontal) to $-\infty$ (when the line is vertical).

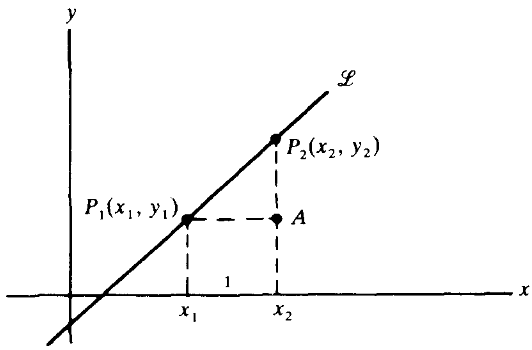


Fig. 3-5

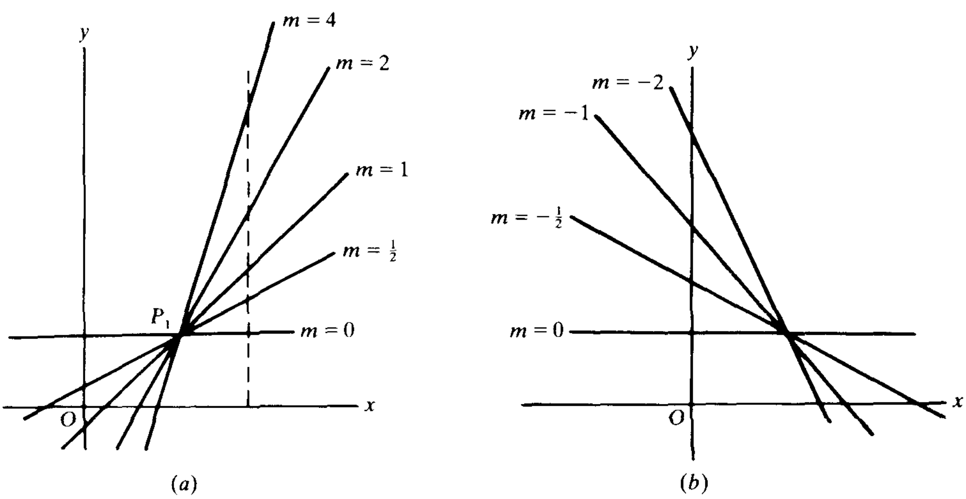
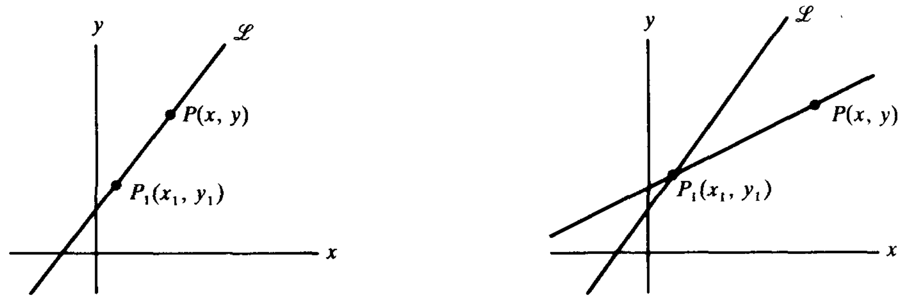


Fig. 3-6

EQUATIONS OF LINES. Let $\mathscr{L}$ be a line that passes through a point $P_1(x_1, y_1)$ and has slope m, as in Fig. 3-7(a). For any other point $P(x, y)$ on the line, the slope m is, by definition, the ratio of $y - y_1$ to $x - x_1$. Thus, for any point (x, y) on $\mathscr{L}$,

$$m = \frac{y - y_1}{x - x_1} \tag{3.1}$$

Conversely, if $P(x, y)$ is *not* on line $\mathscr{L}$, as in Fig. 3-7(b), then the slope $\dfrac{y - y_1}{x - x_1}$ of the line PP_1 is different from the slope m of $\mathscr{L}$; hence (*3.1*) does not hold for points that are not on $\mathscr{L}$. Thus, the line $\mathscr{L}$ consists of only those points (x, y) that satisfy (*3.1*). In such a case, we say that $\mathscr{L}$ is the *graph* of (*3.1*).



(a) (b)

Fig. 3-7

A POINT-SLOPE EQUATION of the line $\mathscr{L}$ is any equation of the form (*3.1*). If the slope m of $\mathscr{L}$ is known, then each point (x_1, y_1) of $\mathscr{L}$ yields a point-slope equation of $\mathscr{L}$. Hence, there are infinitely many point-slope equations for $\mathscr{L}$.

EXAMPLE 2: (*a*) The line passing through the point $(2, 5)$ with slope 3 has a point-slope equation $\frac{y-5}{x-2} = 3$. (*b*) Let $\mathscr{L}$ be the line through the points $(3, -1)$ and $(2, 3)$. Its slope is $m = \frac{3-(-1)}{2-3} = \frac{4}{-1} = -4$. Two point-slope equations of $\mathscr{L}$ are $\frac{y+1}{x-3} = -4$ and $\frac{y-3}{x-2} = -4$.

SLOPE-INTERCEPT EQUATION. If we multiply (*3.1*) by $x - x_1$, we obtain the equation $y - y_1 = m(x - x_1)$, which can be reduced first to $y - y_1 = mx - mx_1$, and then to $y = mx + (y_1 - mx_1)$. Let b stand for the number $y_1 - mx_1$. Then the equation for line $\mathscr{L}$ becomes

$$y = mx + b \tag{3.2}$$

Equation (*3.2*) yields the value $y = b$ when $x = 0$, so the point $(0, b)$ lies on $\mathscr{L}$. Thus, b is the y coordinate of the intersection of $\mathscr{L}$ and the y axis, as shown in Fig. 3-8. The number b is called the *y intercept* of $\mathscr{L}$, and (*3.2*) is called the *slope-intercept equation* for $\mathscr{L}$.

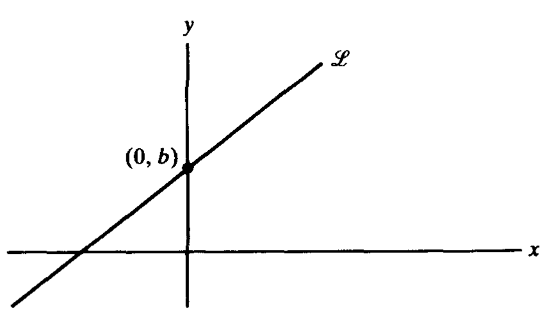


Fig. 3-8

EXAMPLE 3: The line through the points $(2, 3)$ and $(4, 9)$ has slope

$$m = \frac{9-3}{4-2} = \frac{6}{2} = 3$$

Its slope-intercept equation has the form $y = 3x + b$. Since the point $(2, 3)$ lies on the line, $(2, 3)$ must satisfy this equation. Substitution yields $3 = 3(2) + b$, from which we find $b = -3$. Thus, the slope-intercept equation is $y = 3x - 3$.

Another method for finding this equation is to write a point-slope equation of the line, say $\frac{y-3}{x-2} = 3$. Then multiplying by $x - 2$ and adding 3 yield $y = 3x - 3$.

PARALLEL LINES. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be parallel nonvertical lines, and let A_1 and A_2 be the points at which $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect the y axis, as in Fig. 3-9(a). Further, let B_1 be one unit to the right of A_1, and B_2 one unit to the right of A_2. Let C_1 and C_2 be the intersections of the verticals through B_1 and B_2 with $\mathscr{L}_1$ and $\mathscr{L}_2$. Now, triangle $A_1B_1C_1$ is congruent to triangle $A_2B_2C_2$ (by the angle-side-angle congruence theorem). Hence, $\overline{B_1C_1} = \overline{B_2C_2}$ and

$$\text{Slope of } \mathscr{L}_1 = \frac{\overline{B_1C_1}}{1} = \frac{\overline{B_2C_2}}{1} = \text{slope of } \mathscr{L}_2$$

Thus, *parallel lines have equal slopes.*

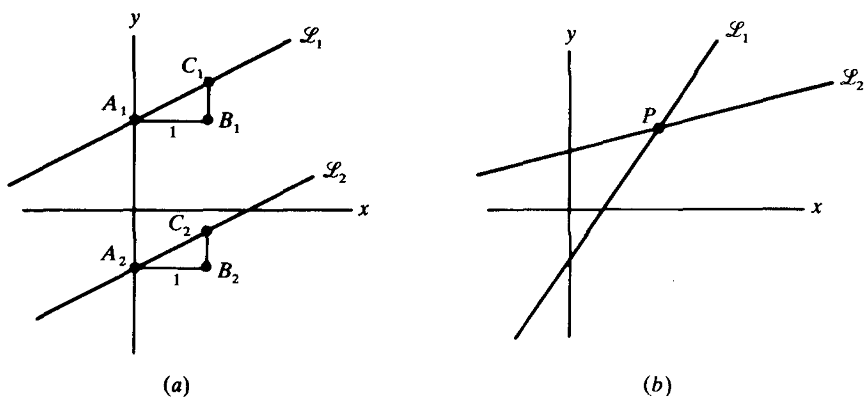


Fig. 3-9

Conversely, assume that two different lines $\mathscr{L}_1$ and $\mathscr{L}_2$ are not parallel, and let them meet at point P, as in Fig. 3-9(b). If $\mathscr{L}_1$ and $\mathscr{L}_2$ had the same slope, then they would have to be the same line. Hence, $\mathscr{L}_1$ and $\mathscr{L}_2$ have different slopes.

Theorem 3.1: Two distinct nonvertical lines are parallel if and only if their slopes are equal.

EXAMPLE 4: Find the slope-intercept equation of the line $\mathscr{L}$ through (4, 1) and parallel to the line $\mathscr{M}$ having the equation $4x - 2y = 5$.

By solving the latter equation for y, we see that $\mathscr{M}$ has the slope-intercept equation $y = 2x - \frac{5}{2}$. Hence, $\mathscr{M}$ has slope 2. The slope of the parallel line $\mathscr{L}$ also must be 2. So the slope-intercept equation of $\mathscr{L}$ has the form $y = 2x + b$. Since (4, 1) lies on $\mathscr{L}$, we can write $1 = 2(4) + b$. Hence, $b = -7$, and the slope-intercept equation of $\mathscr{L}$ is $y = 2x - 7$.

PERPENDICULAR LINES. In Problem 5 we shall prove the following:

Theorem 3.2: Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

If m_1 and m_2 are the slopes of perpendicular lines, then $m_1m_2 = -1$. This is equivalent to $m_2 = -\frac{1}{m_1}$; hence, *the slopes of perpendicular lines are negative reciprocals of each other.*

Solved Problems

1. Find the slope of the line having the equation $3x - 4y = 8$. Draw the line. Do the points $(6, 2)$ and $(12, 7)$ lie on the line?

Solving the equation for y yields $y = \frac{3}{4}x - 2$. This is the slope-intercept equation; the slope is $\frac{3}{4}$ and the y intercept is -2.

Substituting 0 for x shows that the line passes through the point $(0, -2)$. To draw the line, we need another point. If we substitute 4 for x in the slope-intercept equation, we get $y = \frac{3}{4}(4) - 2 = 1$. So, $(4, 1)$ also lies on the line, which is drawn in Fig. 3-10. (We could have found other points on the line by substituting numbers other than 4 for x.)

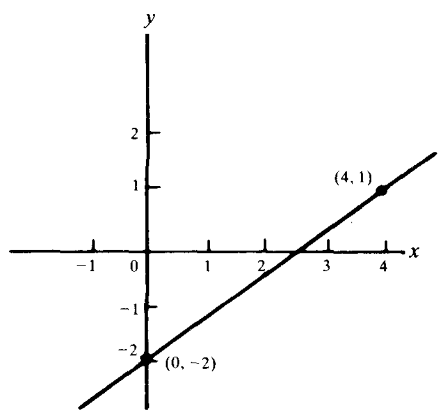


Fig. 3-10

To test whether $(6, 2)$ is on the line, we substitute 6 for x and 2 for y in the original equation, $3x - 4y = 8$. The two sides turn out to be unequal; hence, $(6, 2)$ is not on the line. The same procedure shows that $(12, 7)$ lies on the line.

2. Line $\mathscr{L}$ is the perpendicular bisector of the line segment joining the points $A(-1, 2)$ and $B(3, 4)$, as shown in Fig. 3-11. Find an equation for $\mathscr{L}$.

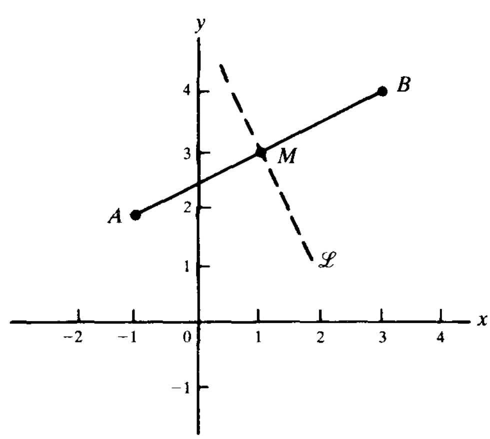


Fig. 3-11

$\mathscr{L}$ passes through the midpoint M of segment AB. By the midpoint formulas (*2.2*), the coordinates of M are $(1, 3)$. The slope of the line through A and B is $\frac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$. Let m be the slope of $\mathscr{L}$. By Theorem 3.2, $\frac{1}{2}m = -1$, whence $m = -2$.

The slope-intercept equation for $\mathscr{L}$ has the form $y = -2x + b$. Since $M\ (1,3)$ lies on $\mathscr{L}$, we have $3 = -2(1) + b$. Hence, $b = 5$, and the slope-intercept equation of $\mathscr{L}$ is $y = -2x + 5$.

3. Determine whether the points $A(1, -1)$, $B(3, 2)$, and $C(7, 8)$ are collinear, that is, lie on the same line.

A, B, and C are collinear if and only if the line AB is identical with the line AC, which is equivalent to the slope of AB being equal to the slope of AC. (Why?) The slopes of AB and AC are $\frac{2-(-1)}{3-1} = \frac{3}{2}$ and $\frac{8-(-1)}{7-1} = \frac{9}{6} = \frac{3}{2}$. Hence, A, B, and C are collinear.

4. Prove analytically that the figure obtained by joining the midpoints of consecutive sides of a quadrilateral is a parallelogram.

Locate a quadrilateral with consecutive vertices A, B, C, and D on a coordinate system so that A is the origin, B lies on the positive x axis, and C and D lie above the x axis. (See Fig. 3-12.) Let b be the x coordinate of B, (u, v) the coordinates of C, and (x, y) the coordinates of D. Then, by the midpoint formula (*2.2*), the midpoints M_1, M_2, M_3, and M_4 of sides AB, BC, CD, and DA have coordinates $\left(\frac{b}{2}, 0\right)$, $\left(\frac{u+b}{2}, \frac{v}{2}\right)$, $\left(\frac{x+u}{2}, \frac{y+v}{2}\right)$, and $\left(\frac{x}{2}, \frac{y}{2}\right)$, respectively. We must show that $M_1M_2M_3M_4$ is a parallelogram. To do this, it suffices to prove that lines M_1M_2 and M_3M_4 are parallel and that lines M_2M_3 and M_1M_4 are parallel. Let us calculate the slopes of these lines:

$$\text{Slope}(M_1M_2) = \frac{\frac{v}{2} - 0}{\frac{u+b}{2} - \frac{b}{2}} = \frac{\frac{v}{2}}{\frac{u}{2}} = \frac{v}{u} \qquad \text{slope}(M_3M_4) = \frac{\frac{y}{2} - \frac{y+v}{2}}{\frac{x}{2} - \frac{x+u}{2}} = \frac{-\frac{v}{2}}{-\frac{u}{2}} = \frac{v}{u}$$

$$\text{Slope}(M_2M_3) = \frac{\frac{y+v}{2} - \frac{v}{2}}{\frac{x+u}{2} - \frac{u+b}{2}} = \frac{\frac{y}{2}}{\frac{x-b}{2}} = \frac{y}{x-b} \qquad \text{slope}(M_1M_4) = \frac{\frac{y}{2} - 0}{\frac{x}{2} - \frac{b}{2}} = \frac{y}{x-b}$$

Since $\text{slope}(M_1M_2) = \text{slope}(M_3M_4)$, M_1M_2 and M_3M_4 are parallel. Since $\text{slope}(M_2M_3) = \text{slope}(M_1M_4)$, M_2M_3 and M_1M_4 are parallel. Thus, $M_1M_2M_3M_4$ is a parallelogram.

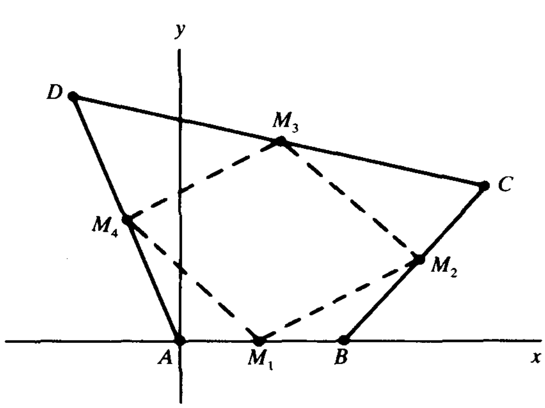


Fig. 3-12

5. Prove Theorem 3.2.

First we assume $\mathscr{L}_1$ and $\mathscr{L}_2$ are perpendicular nonvertical lines with slopes m_1 and m_2. We must show that $m_1 m_2 = -1$. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be the lines through the origin O that are parallel to $\mathscr{L}_1$ and $\mathscr{L}_2$, as shown in Fig. 3-13(*a*). Then the slope of $\mathscr{M}_1$ is m_1, and the slope of $\mathscr{M}_2$ is m_2 (by Theorem I). Moreover, $\mathscr{M}_1$ and $\mathscr{M}_2$ are perpendicular, since $\mathscr{L}_1$ and $\mathscr{L}_2$ are perpendicular.

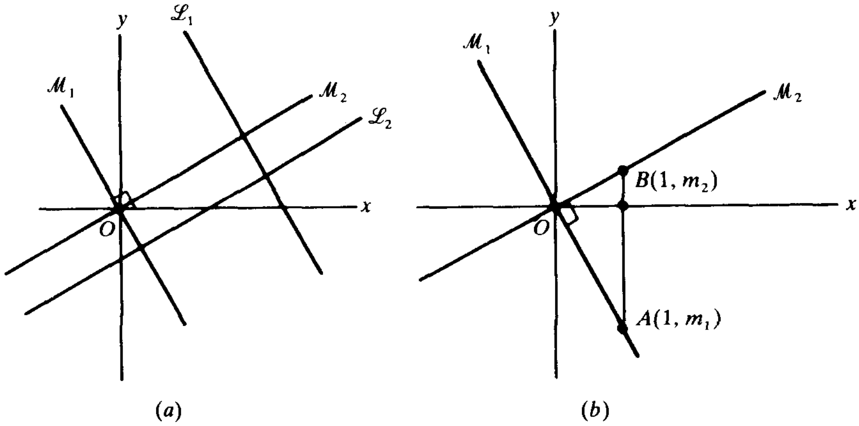


Fig. 3-13

Now let A be the point on $\mathscr{M}_1$ with x coordinate 1, and let B be the point on $\mathscr{M}_2$ with x coordinate 1, as in Fig. 3-13(*b*). The slope-intercept equation of $\mathscr{M}_1$ is $y = m_1 x$; therefore, the y coordinate of A is m_1, since its x coordinate is 1. Similarly, the y coordinate of B is m_2. By the distance formula (*2.1*),

$$\overline{OB} = \sqrt{(1-0)^2 + (m_2 - 0)^2} = \sqrt{1 + m_2^2}$$
$$\overline{OA} = \sqrt{(1-0)^2 + (m_1 - 0)^2} = \sqrt{1 + m_1^2}$$
$$\overline{BA} = \sqrt{(1-1)^2 + (m_2 - m_1)^2} = \sqrt{(m_2 - m_1)^2}$$

Then by the Pythagorean theorem for right triangle BOA,

$$\overline{BA}^2 = \overline{OB}^2 + \overline{OA}^2$$

or

$$(m_2 - m_1)^2 = (1 + m_2^2) + (1 + m_1^2)$$
$$m_2^2 - 2m_2 m_1 + m_1^2 = 2 + m_2^2 + m_1^2$$
$$m_2 m_1 = -1$$

Now, conversely, we assume that $m_1 m_2 = -1$, where m_1 and m_2 are the slopes of nonvertical lines $\mathscr{L}_1$ and $\mathscr{L}_2$. Then $\mathscr{L}_1$ is not parallel to $\mathscr{L}_2$. (Otherwise, by Theorem 3.1, $m_1 = m_2$ and, therefore,

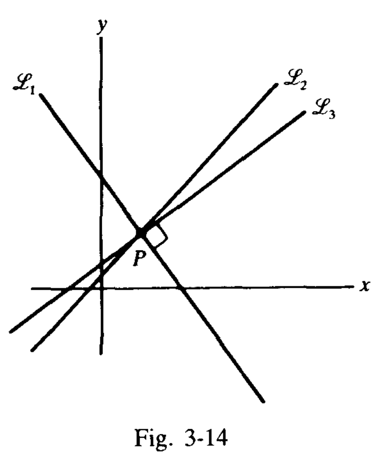


Fig. 3-14

$m_1^2 = -1$, which contradicts the fact that the square of a real number is never negative.) We must show that $\mathscr{L}_1$ and $\mathscr{L}_2$ are perpendicular. Let P be the intersection of $\mathscr{L}_1$ and $\mathscr{L}_2$ (see Fig. 3-14). Let $\mathscr{L}_3$ be the line through P that is perpendicular to $\mathscr{L}_1$. If m_3 is the slope of $\mathscr{L}_3$, then, by the first part of the proof, $m_1m_3 = -1$ and, therefore, $m_1m_3 = m_1m_2$. Since $m_1m_3 = -1$, $m_1 \neq 0$; therefore, $m_3 = m_2$. Since $\mathscr{L}_2$ and $\mathscr{L}_3$ pass through the same point P and have the same slope, they must coincide. Since $\mathscr{L}_1$ and $\mathscr{L}_3$ are perpendicular, $\mathscr{L}_1$ and $\mathscr{L}_2$ are also perpendicular.

6. Show that, if a and b are not both zero, then the equation $ax + by = c$ is the equation of a line and, conversely, every line has an equation of that form.

Assume $b \neq 0$. Then, if the equation $ax + by = c$ is solved for y, we obtain a slope-intercept equation $y = (-a/b)x + c/b$ of a line. If $b = 0$, then $a \neq 0$, and the equation $ax + by = c$ reduces to $ax = c$; this is equivalent to $x = c/a$, the equation of a vertical line.

Conversely, every nonvertical line has a slope-intercept equation $y = mx + b$, which is equivalent to $-mx + y = b$, an equation of the desired form. A vertical line has an equation of the form $x = c$, which is also an equation of the required form with $a = 1$ and $b = 0$.

7. Show that the line $y = x$ makes an angle of 45° with the positive x axis (that is, that angle BOA in Fig. 3-15 contains 45°).

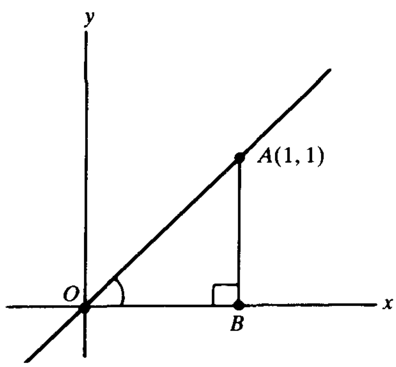


Fig. 3-15

Let A be the point on the line $y = x$ with coordinates $(1, 1)$. Drop a perpendicular AB to the positive x axis. Then $\overline{AB} = 1$ and $\overline{OB} = 1$. Hence, angle OAB = angle BOA, since they are the base angles of isosceles triangle BOA. Since angle OBA is a right angle,

$$\text{Angle } OAB + \text{angle } BOA = 180° - \text{angle } OBA = 180° - 90° = 90°$$

Since angle BOA = angle OAB, they each contain 45°.

8. Show that the distance d from a point $P(x_1, y_1)$ to a line $\mathscr{L}$ with equation $ax + by = c$ is given by the formula $d = \dfrac{|ax + by - c|}{\sqrt{a^2 + b^2}}$.

Let $\mathscr{M}$ be the line through P that is perpendicular to $\mathscr{L}$. Then $\mathscr{M}$ intersects $\mathscr{L}$ at some point Q with coordinates (u, v), as in Fig. 3-16. Clearly, d is the length $\overline{PQ}$, so if we can find u and v, we can compute d with the distance formula. The slope of $\mathscr{L}$ is $-a/b$. Hence, by Theorem 3.2, the slope of $\mathscr{M}$ is b/a. Then a point-slope equation of $\mathscr{M}$ is $\dfrac{y - y_1}{x - x_1} = \dfrac{b}{a}$. Thus, u and v are the solutions of the pair of equations $au + bv = c$ and $\dfrac{v - y_1}{u - x_1} = \dfrac{b}{a}$. Tedious algebraic calculations yield the solution

$$u = \frac{ac + b^2x_1 + aby_1}{a^2 + b^2} \qquad \text{and} \qquad v = \frac{bc - abx_1 + a^2y_1}{a^2 + b^2}$$

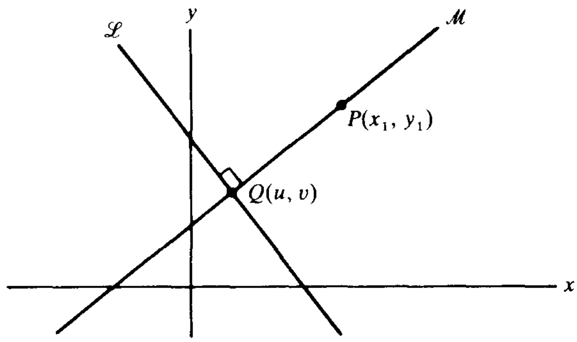


Fig. 3-16

The distance formula, together with further calculations, yields

$$d = \overline{PQ} = \sqrt{(x_1 - u)^2 + (y_1 - v)^2} = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$$

Supplementary Problems

9. Find a point-slope equation for the line through each of the following pairs of points: (*a*) (3, 6) and (2, −4); (*b*) (8, 5) and (4, 0); (*c*) (1, 3) and the origin; (*d*) (2, 4) and (−2, 4).

Ans. (*a*) $\frac{y-6}{x-3} = 10$; (*b*) $\frac{y-5}{x-8} = \frac{5}{4}$; (*c*) $\frac{y-3}{x-1} = 3$; (*d*) $\frac{y-4}{x-2} = 0$

10. Find the slope-intercept equation of each line:
(*a*) Through the points (4, −2) and (1, 7)
(*b*) Having slope 3 and y intercept 4
(*c*) Through the points (−1, 0) and (0, 3)
(*d*) Through (2, −3) and parallel to the x axis
(*e*) Through (2, 3) and rising 4 units for every unit increase in x
(*f*) Through (−2, 2) and falling 2 units for every unit increase in x
(*g*) Through (3, −4) and parallel to the line with equation $5x - 2y = 4$
(*h*) Through the origin and parallel to the line with equation $y = 2$
(*i*) Through (−2, 5) and perpendicular to the line with equation $4x + 8y = 3$
(*j*) Through the origin and perpendicular to the line with equation $3x - 2y = 1$
(*k*) Through (2, 1) and perpendicular to the line with equation $x = 2$
(*l*) Through the origin and bisecting the angle between the positive x axis and the positive y axis

Ans. (*a*) $y = -3x + 10$; (*b*) $y = 3x + 3$; (*c*) $y = 3x + 3$; (*d*) $y = -3$; (*e*) $y = 4x - 5$; (*f*) $y = -2x - 2$; (*g*) $y = \frac{5}{2}x - \frac{23}{2}$; (*h*) $y = 0$; (*i*) $y = 2x + 9$; (*j*) $y = -\frac{2}{3}x$; (*k*) $y = 1$; (*l*) $y = x$

11. (*a*) Describe the lines having equations of the form $x = a$.
(*b*) Describe the lines having equations of the form $y = b$.
(*c*) Describe the line having the equation $y = -x$.

12. (*a*) Find the slopes and y intercepts of the lines that have the following equations: (i) $y = 3x - 2$; (ii) $2x - 5y = 3$; (iii) $y = 4x - 3$; (iv) $y = -3$; (v) $\frac{y}{2} + \frac{x}{3} = 1$.
(*b*) Find the coordinates of a point other than $(0, b)$ on each of the lines of part (*a*).

Ans. (*a*) (i) $m = 3$, $b = -2$; (ii) $m = \frac{2}{5}$, $b = -\frac{3}{5}$; (iii) $m = 4$, $b = -3$; (iv) $m = 0$, $b = -3$; (v) $m = -\frac{2}{3}$, $b = 2$. (*b*) (i) $(1, 1)$; (ii) $(-6, -3)$; (iii) $(1, 1)$; (iv) $(1, -3)$; (v) $(3, 0)$

13. If the point $(3, k)$ lies on the line with slope $m = -2$ passing through the point $(2, 5)$, find k.

Ans. $k = 3$

14. Does the point $(3, -2)$ lie on the line through the points $(8, 0)$ and $(-7, -6)$?

Ans. yes

15. Use slopes to determine whether the points $(7, -1)$, $(10, 1)$, and $(6, 7)$ are the vertices of a right triangle.

Ans. They are.

16. Use slopes to determine whether $(8, 0)$, $(-1, -2)$, $(-2, 3)$, and $(7, 5)$ are the vertices of a parallelogram.

Ans. They are.

17. Under what conditions are the points $(u, v + w)$, $(v, u + w)$, and $(w, u + v)$ collinear?

Ans. always

18. Determine k so that the points $A(7, 3)$, $B(-1, 0)$, and $C(k, -2)$ are the vertices of a right triangle with right angle at B.

Ans. $k = 1$

19. Determine whether the following pairs of lines are parallel, perpendicular, or neither:
(*a*) $y = 3x + 2$ and $y = 3x - 2$
(*b*) $y = 2x - 4$ and $y = 3x + 5$
(*c*) $3x - 2y = 5$ and $2x + 3y = 4$
(*d*) $6x + 3y = 1$ and $4x + 2y = 3$
(*e*) $x = 3$ and $y = -4$
(*f*) $5x + 4y = 1$ and $4x + 5y = 2$
(*g*) $x = -2$ and $x = 7$.

Ans. (*a*) parallel; (*b*) neither; (*c*) perpendicular; (*d*) parallel; (*e*) perpendicular; (*f*) neither; (*g*) parallel

20. Draw the lines determined by the equation $2x + 5y = 10$. Determine if the points $(10, 2)$ and $(12, 3)$ lie on this line.

21. For what values of k will the line $kx - 3y = 4k$ have the following properties: (*a*) have slope 1; (*b*) have y intercept 2; (*c*) pass through the point $(2, 4)$; (*d*) be parallel to the line $2x - 4y = 1$; (*e*) be perpendicular to the line $x - 6y = 2$?

Ans. (*a*) $k = 3$; (*b*) $k = -\frac{3}{2}$; (*c*) $k = -6$; (*d*) $k = \frac{3}{2}$; (*e*) $k = -18$

22. Describe geometrically the families of lines (*a*) $y = mx - 3$ and (*b*) $y = 4x + b$, where m and b are any real numbers.

Ans. (*a*) lines with y intercept -3; (*b*) lines with slope 4

23. In the triangle with vertices $A(0, 0)$, $B(2, 0)$, and $C(3, 3)$, find equations for (*a*) the median from B to the midpoint of the opposite side; (*b*) the perpendicular bisector of side BC; and (c) the altitude from B to the opposite side.

Ans. (*a*) $y = -3x + 6$; (*b*) $x + 3y = 7$; (*c*) $y = -x + 2$

24. In the triangle with vertices $A(2,0)$, $B(1,6)$, and $C(3,9)$, find the slope-intercept equation of (*a*) the median from B to the opposite side; (*b*) the perpendicular bisector of side AB; (*c*) the altitude from A to the opposite side.

Ans. (*a*) $y = -x + 7$; (*b*) $y = \frac{1}{6}x + \frac{11}{4}$; (*c*) $y = -\frac{2}{3}x + \frac{4}{3}$

25. Temperature is usually measured in either Fahrenheit or Celsius degrees. Fahrenheit (F) and Celsius (C) temperatures are related by a linear equation of the form $F = aC + b$. The freezing point of water is 0°C and 32°F, and the boiling point of water is 100°C and 212°F. (*a*) Find the equation relating F and C. (*b*) What temperature is the same in both scales?

Ans. (*a*) $F = \frac{9}{5}C + 32$; (*b*) $-40°$

26. The x *intercept* of a line $\mathscr{L}$ is defined to be the x coordinate of the unique point where $\mathscr{L}$ intersects the x axis. It is the number a for which $(a, 0)$ lies on $\mathscr{L}$.

(*a*) Which lines do not have x intercepts?

(*b*) Find the x intercepts of (i) $3x - 4y = 2$; (ii) $x + y = 1$; (iii) $12x - 13y = 2$; (iv) $x = 2$; (v) $y = 0$.

(*c*) If a and b are the x intercept and y intercept of a line, show that $x/a + y/b = 1$ is an equation of the line.

(*d*) If $x/a + y/b = 1$ is an equation of a line, show that a and b are the x intercept and y intercept of the line.

Ans. (*a*) horizontal lines. (*b*) (i) $\frac{2}{3}$; (ii) 1; (iii) $\frac{1}{6}$; (iv) 2; (v) none

27. Prove analytically that the diagonals of a rhombus (a parallelogram of which all sides are equal) are perpendicular to each other.

28. (*a*) Prove analytically that the altitudes of a triangle meet at a point. [*Hint*: Let the vertices of the triangle be $(2a, 0)$, $(2b, 0)$ and $(0, 2c)$.]

(*b*) Prove analytically that the medians of a triangle meet at a point (called the *centroid*).

(*c*) Prove analytically that the perpendicular bisectors of the sides of a triangle meet at a point.

(*d*) Prove that the three points in parts (*a*) to (*c*) are collinear.

29. Prove analytically that a parallelogram with perpendicular diagonals is a rhombus.

30. Prove analytically that a quadrilateral with diagonals that bisect each other is a parallelogram.

31. Prove analytically that the line joining the midpoints of two sides of a triangle is parallel to the third side.

32. (*a*) If a line $\mathscr{L}$ has the equation $5x + 3y = 4$, prove that a point $P(x, y)$ is above $\mathscr{L}$ if and only if $5x + 3y > 4$.

(*b*) If a line $\mathscr{L}$ has the equation $ax + by = c$ and $b > 0$, prove that a point $P(x, y)$ is above $\mathscr{L}$ if and only if $ax + by > c$.

(*c*) If a line $\mathscr{L}$ has the equation $ax + by = c$ and $b < 0$, prove that a point $P(x, y)$ is above $\mathscr{L}$ if and only if $ax + by < c$.

33. Use two inequalities to describe the set of all points above the line $3x + 2y = 7$ and below the line $4x - 2y = 1$. Draw a diagram showing the set.

Ans. $3x + 2y > 7$; $4x - 2y < 1$

34. Find the distance from the point $(4, 7)$ to the line $3x + 4y = 1$.

Ans. $\frac{39}{5}$

35. Find the distance from the point $(-1, 2)$ to the line $8x - 15y = 3$.

Ans. $\frac{41}{17}$

36. Find the area of the triangle with vertices $A(0, 1)$, $B(5, 3)$, and $C(2, -2)$.

Ans. $\frac{19}{2}$

37. Show that two equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ determine parallel lines if and only if $a_1b_2 = a_2b_1$. (When neither a_2 nor b_2 is 0, this is equivalent to $a_1/a_2 = b_1/b_2$.)

38. Show that two equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ determine the same line if and only if the coefficients of one equation are proportional to those of the other, that is, there is a number r such that $a_1 = ra_2$, $b_1 = rb_2$, and $c_1 = rc_2$.

39. If $ax + by = c$ is an equation of a line $\mathscr{L}$ and $c \geq 0$, then the *normal equation* of $\mathscr{L}$ is defined to be

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{c}{\sqrt{a^2+b^2}}$$

(*a*) Show that $|c|/\sqrt{a^2+b^2}$ is the distance from the origin to $\mathscr{L}$.
(*b*) Find the normal equation of the line $5x - 12y = 26$ and compute the distance from the origin to the line.

Ans. (*b*) $\frac{5}{13}x - \frac{12}{13}y = 2$; distance $= 2$

40. Find equations of the lines parallel to the line $3x + 4y = 7$ and at a perpendicular distance of 4 from it.

Ans. $3x + 4y = -13$; $3x + 4y = 27$

41. Show that a point-slope equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$.

42. Find the values of k such that the distance from $(-2, 3)$ to the line $7x - 24y = k$ is 3.

Ans. $k = -11$; $k = -161$

43. Find equations for the families of lines (*a*) passing through $(2, 5)$; (*b*) having slope 3; (*c*) having y intercept 1; (*d*) having x intercept -2; (*e*) having y intercept three times the x intercept; (*f*) whose x intercept and y intercept add up to 6.

Ans. (*a*) $y - 5 = m(x - 2)$; (*b*) $y = 3x + b$; (*c*) $y = mx + 1$; (*d*) $y = m(x + 2)$; (*e*) $3x + y = 3a$; (*f*) $\dfrac{x}{a} + \dfrac{y}{6 - a} = 1$

44. Find the value of k such that the line $3x - 4y = k$ determines, with the coordinate axes, a triangle of area 6.

Ans. $k = \pm 12$

45. Find the point on the line $3x + y = -4$ that is equidistant from $(-5, 6)$ and $(3, 2)$.

Ans. $(-2, 2)$

46. Find the equation of the line that passes through the point of intersection of the lines $3x - 2y = 6$ and $x + 3y = 13$ and whose distance from the origin is 5.

Ans. $4x + 3y = 25$

47. Find the equations of the two lines that are the bisectors of the angles formed by the intersection of the lines $3x + 4y = 2$ and $5x - 12y = 7$. (*Hint*: Points on an angle bisector are equidistant from the two sides.)

Ans. $14x + 112y + 9 = 0$; $64x - 8y - 61 = 0$

48. (*a*) Find the distance between the parallel lines $3x + 4y = 2$ and $6x + 8y = 1$. (*b*) Find the equation of the line midway between the lines of part (*a*).

Ans. (*a*) $\frac{3}{10}$; (*b*) $12x + 16y = 5$

49. What are the conditions on a, b, and c so that the line $ax + by = c$ forms an isosceles triangle with the coordinate axes?

Ans. $|a| = |b|$

50. Show that, if a, b, and c are nonzero, the area bounded by the line $ax + by = c$ and the coordinate axes is $\frac{1}{2}c^2/|ab|$.

51. Show that the lines $ax + by = c_1$ and $bx - ay = c_2$ are perpendicular.

52. Show that the area of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ is $\frac{1}{2}[(x_1 - x_2)(y_2 - y_3) - (y_1 - y_2)(x_2 - x_3)]$. (*Hint*: The altitude from A to side BC is the distance from A to the line through B and C.)

53. Show that the distance between parallel lines $ax + by = c_1$ and $ax + by = c_2$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

54. Prove that, if the lines $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are nonparallel lines that intersect at point P, then, for any number k, the equation $(a_1x + b_1y - c_1) + k(a_2x + b_2y - c_2) = 0$ determines a line through P. Conversely, any line through P other than $a_2x + b_2y = c_2$ is represented by such an equation for a suitable value of k.

55. Of all the lines that pass through the intersection point of the two lines $2x - 3y = 5$ and $4x + y = 2$, find an equation of the line that also passes through $(1, 0)$.

Ans. $16x - 3y = 16$

Chapter 4

Circles

EQUATIONS OF CIRCLES. For a point $P(x, y)$ to lie on the circle with center $C(a, b)$ and radius r, the distance $\overline{PC}$ must be equal to r (see Fig. 4-1). By the distance formula (*2.1*),

$$\overline{PC} = \sqrt{(x-a)^2 + (y-b)^2}$$

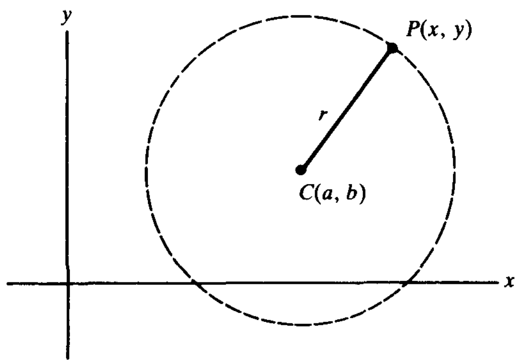


Fig. 4-1

Thus, P lies on the circle if and only if

$$(x-a)^2 + (y-b)^2 = r^2 \tag{4.1}$$

Equation (*4.1*) is called the *standard equation* of the circle with center at (a, b) and radius r.

EXAMPLE 1: (*a*) The circle with center $(3, 1)$ and radius 2 has the equation $(x-3)^2 + (y-1)^2 = 4$.
(*b*) The circle with center $(2, -1)$ and radius 3 has the equation $(x-2)^2 + (y+1)^2 = 9$.
(*c*) What is the set of points satisfying the equation $(x-4)^2 + (y-5)^2 = 25$?
By (*4.1*), this is the equation of the circle with center at $(4, 5)$ and radius 5. That circle is said to be the *graph* of the given equation, that is, the set of points satisfying the equation.
(*d*) The graph of the equation $(x+3)^2 + y^2 = 2$ is the circle with center at $(-3, 0)$ and radius $\sqrt{2}$.

THE STANDARD EQUATION OF A CIRCLE with center at the origin $(0, 0)$ and radius r is

$$x^2 + y^2 = r \tag{4.2}$$

For example, $x^2 + y^2 = 1$ is the equation of the circle with center at the origin and radius 1. The graph of $x^2 + y^2 = 5$ is the circle with center at the origin and radius $\sqrt{5}$.

The equation of a circle sometimes appears in a disguised form. For example, the equation

$$x^2 + y^2 + 8x - 6y + 21 = 0 \tag{4.3}$$

turns out to be equivalent to

$$(x+4)^2 + (y-3)^2 = 4 \tag{4.4}$$

Equation (*4.4*) is the standard equation of a circle with center at $(-4, 3)$ and radius 2.

Equation (*4.4*) is obtained from (*4.3*) by a process called *completing the square*. In general terms, the process involves finding the number that must be added to the sum $x^2 + Ax$ to obtain a square. Here, we note that $\left(x + \frac{A}{2}\right)^2 = x^2 + Ax + \left(\frac{A}{2}\right)^2$. Thus, in general, *we must add* $\left(\frac{A}{2}\right)^2$ *to* $x^2 + Ax$ *to obtain the square* $\left(x + \frac{A}{2}\right)^2$. For example, to get a square from $x^2 + 8x$, we add $(\frac{8}{2})^2$, that is, 16. The result is $x^2 + 8x + 16$, which is $(x + 4)^2$. This is the process of completing the square.

Consider the original (*4.3*): $x^2 + y^2 + 8x - 6y + 21 = 0$. To complete the square in $x^2 + 8x$, we add 16. To complete the square in $y^2 - 6y$, we add $(-\frac{6}{2})^2$, which is 9. Of course, since we added 16 and 9 to the left side of the equation, we must also add them to the right side, obtaining

$$(x^2 + 8x + 16) + (y^2 - 6y + 9) + 21 = 16 + 9$$

This is equivalent to

$$(x + 4)^2 + (y - 3)^2 + 21 = 25$$

and subtraction of 21 from both sides yields (*4.4*).

EXAMPLE 2: Consider the equation $x^2 + y^2 - 4x - 10y + 20 = 0$. Completing the square yields

$$(x^2 - 4x + 4) + (y^2 - 10y + 25) + 20 = 4 + 25$$
$$(x - 2)^2 + (y - 5)^2 = 9$$

Thus, the original equation is the equation of a circle with center at $(2, 5)$ and radius 3.

The process of completing the square can be applied to any equation of the form

$$x^2 + y^2 + Ax + By + C = 0 \tag{4.5}$$

to obtain

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 + C = \frac{A^2}{4} + \frac{B^2}{4}$$

or

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2 + B^2 - 4C}{4} \tag{4.6}$$

There are three different cases, depending on whether $A^2 + B^2 - 4C$ is positive, zero, or negative.

Case 1: $A^2 + B^2 - 4C > 0$. In this case, (*4.6*) is the standard equation of a circle with center at $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ and radius $\frac{\sqrt{A^2 + B^2 - 4C}}{2}$.

Case 2: $A^2 + B^2 - 4C = 0$. A sum of the squares of two quantities is zero when and only when each of the quantities is zero. Hence, (*4.6*) is equivalent to the conjunction of the equations $x + A/2 = 0$ and $y + B/2 = 0$ in this case, and the only solution of (*4.6*) is the point $(-A/2, -B/2)$. Hence, the graph of (4.5) is a single point, which may be considered a *degenerate circle* of radius 0.

Case 3: $A^2 + B^2 - 4C < 0$. A sum of two squares cannot be negative. So, in this case, (*4.5*) has no solution at all.

We can show that any circle has an equation of the form (*4.5*). Suppose its center is (a, b) and its radius is r; then its standard equation is

$$(x - a)^2 + (y - b)^2 = r^2$$

Expanding yields $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$, or

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

Solved Problems

1. Identify the graphs of (a) $2x^2 + 2y^2 - 4x + y + 1 = 0$; ($b$) $x^2 + y^2 - 4y + 7 = 0$; (c) $x^2 + y^2 - 6x - 2y + 10 = 0$.

(a) First divide by 2, obtaining $x^2 + y^2 - 2x + \frac{1}{2}y + \frac{1}{2} = 0$. Then complete the squares:

$$(x^2 - 2x + 1) + (y^2 + \tfrac{1}{2}y + \tfrac{1}{16}) + \tfrac{1}{2} = 1 + \tfrac{1}{16} = \tfrac{17}{16}$$
$$(x-1)^2 + (y + \tfrac{1}{4})^2 = \tfrac{17}{16} - \tfrac{1}{2} = \tfrac{17}{16} - \tfrac{8}{16} = \tfrac{9}{16}$$

Thus, the graph is the circle with center $(1, -\frac{1}{4})$ and radius $\frac{3}{4}$.

(b) Complete the square:

$$x^2 + (y-2)^2 + 7 = 4$$
$$x^2 + (y-2)^2 = -3$$

Because the right side is negative, there are no points in the graph.

(c) Complete the square:

$$(x-3)^2 + (y-1)^2 + 10 = 9 + 1$$
$$(x-3)^2 + (y-1)^2 = 0$$

The only solution is the point (3, 1).

2. Find the standard equation of the circle with center at $C(2, 3)$ and passing through the point $P(-1, 5)$.

The radius of the circle is the distance

$$\overline{CP} = \sqrt{(5-3)^2 + (-1-2)^2} = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$$

so the standard equation is $(x-2)^2 + (y-3)^2 = 13$.

3. Find the standard equation of the circle passing through the points $P(3, 8)$, $Q(9, 6)$, and $R(13, -2)$.

First method: The circle has an equation of the form $x^2 + y^2 + Ax + By + C = 0$. Substitute the values of x and y at point P, to obtain $9 + 64 + 3A + 8B + C = 0$ or

$$3A + 8B + C = -73 \qquad (1)$$

A similar procedure for points Q and R yields the equations

$$9A + 6B + C = -117 \qquad (2)$$
$$13A - 2B + C = -173 \qquad (3)$$

Eliminate C from (1) and (2) by subtracting (2) from (1):

$$-6A + 2B = 44 \qquad \text{or} \qquad -3A + B = 22 \qquad (4)$$

Eliminate C from (1) and (3) by subtracting (3) from (1):

$$-10A + 10B = 100 \qquad \text{or} \qquad -A + B = 10 \qquad (5)$$

Eliminate B from (4) and (5) by subtracting (5) from (4), obtaining $A = -6$. Substitute this value in (5) to find that $B = 4$. Then solve for C in (1): $C = -87$.

Hence, the original equation for the circle is $x^2 + y^2 - 6x + 4y - 87 = 0$. Completing the squares then yields

$$(x-3)^2 + (y+2)^2 = 87 + 9 + 4 = 100$$

Thus, the circle has center $(3, -2)$ and radius 10.

Second method: The perpendicular bisector of any chord of a circle passes through the center of the circle. Hence, the perpendicular bisector $\mathscr{L}$ of chord PQ will intersect the perpendicular bisector $\mathscr{M}$ of chord QR at the center of the circle (see Fig. 4-2).

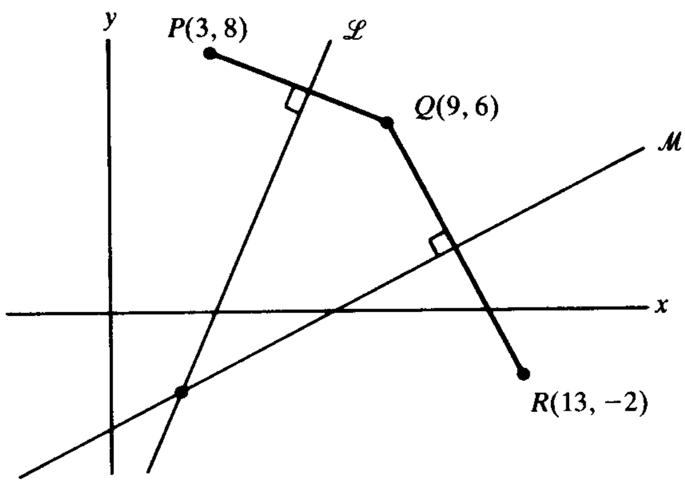


Fig. 4-2

The slope of line PQ is $-\frac{1}{3}$. So, by Theorem 3.2, the slope of $\mathscr{L}$ is 3. Also, $\mathscr{L}$ passes through the midpoint $(6,7)$ of segment PQ. Hence a point-slope equation of $\mathscr{L}$ is $\frac{y-7}{x-6}=3$, and therefore its slope-intercept equation is $y=3x-11$. Similarly, the slope of line QR is -2, and therefore the slope of $\mathscr{M}$ is $\frac{1}{2}$. Since $\mathscr{M}$ passes through the midpoint $(11,2)$ of segment QR, it has a point-slope equation $\frac{y-2}{x-11}=\frac{1}{2}$, which yields the slope-intercept equation $y=\frac{1}{2}x-\frac{7}{2}$. Hence, the coordinates of the center of the circle satisfy the two equations $y=3x-11$ and $y=\frac{1}{2}x-\frac{7}{2}$, and we may write

$$3x-11=\tfrac{1}{2}x-\tfrac{7}{2}$$

from which we find that $x=3$. Therefore,

$$y=3x-11=3(3)-11=-2$$

So the center is at $(3,-2)$. The radius is the distance between the center and the point $(3,8)$:

$$\sqrt{(-2-8)^2+(3-3)^2}=\sqrt{(-10)^2}=\sqrt{100}=10$$

Thus, the standard equation of the circle is $(x-3)^2+(y+2)^2=100$.

4. **Find the center and radius of the circle that passes through $P(1,1)$ and is tangent to the line $y=2x-3$ at the point $Q(3,3)$. (See Fig. 4-3.)**

The line $\mathscr{L}$ perpendicular to $y=2x-3$ at $(3,3)$ must pass through the center of the circle. By Theorem 3.2, the slope of $\mathscr{L}$ is $-\frac{1}{2}$. Therefore, the slope-intercept equation of $\mathscr{L}$ has the form $y=-\frac{1}{2}x+b$. Since $(3,3)$ is on $\mathscr{L}$, we have $3=-\frac{1}{2}(3)+b$; hence, $b=\frac{9}{2}$, and $\mathscr{L}$ has the equation $y=-\frac{1}{2}x+\frac{9}{2}$.

The perpendicular bisector $\mathscr{M}$ of chord PQ in Fig. 4-3 also passes through the center of the circle, so the intersection of $\mathscr{L}$ and $\mathscr{M}$ will be the center of the circle. The slope of $\overline{PQ}$ is 1. Hence, by Theorem 3.2, the slope of $\mathscr{M}$ is -1. So $\mathscr{M}$ has the slope-intercept equation $y=-x+b'$. Since the midpoint $(2,2)$ of chord PQ is a point on $\mathscr{M}$, we have $2=-(2)+b'$; hence, $b'=4$, and the equation of $\mathscr{M}$ is $y=-x+4$. We must find the common solution of $y=-x+4$ and $y=-\frac{1}{2}x+\frac{9}{2}$. Setting

$$-x+4=-\tfrac{1}{2}x+\tfrac{9}{2}$$

yields $x=-1$. Therefore, $y=-x+4=-(-1)+4=5$, and the center C of the circle is $(-1,5)$. The radius is the distance $\overline{PC}=\sqrt{(-1-3)^2+(5-3)^2}=\sqrt{16+4}=\sqrt{20}$. The standard equation of the circle is then $(x+1)^2+(y-5)^2=20$.

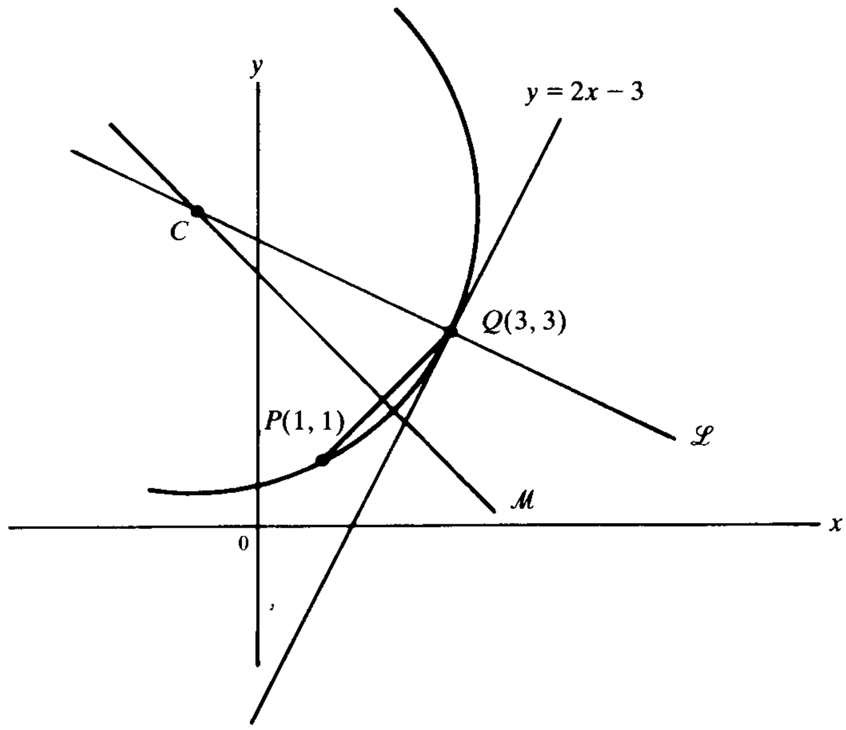


Fig. 4-3

5. Find the standard equation of every circle that passes through the points $P(1, -1)$ and $Q(3, 1)$ and is tangent to the line $y = -3x$.

Let $C(c, d)$ be the center of one of the circles, and let A be the point of tangency (see Fig. 4-4). Then, because $\overline{CP} = \overline{CQ}$, we have

$$\overline{CP}^2 = \overline{CQ}^2 \qquad \text{or} \qquad (c-1)^2 + (d+1)^2 = (c-3)^2 + (d-1)^2$$

Expanding and simplifying, we obtain

$$c + d = 2 \tag{1}$$

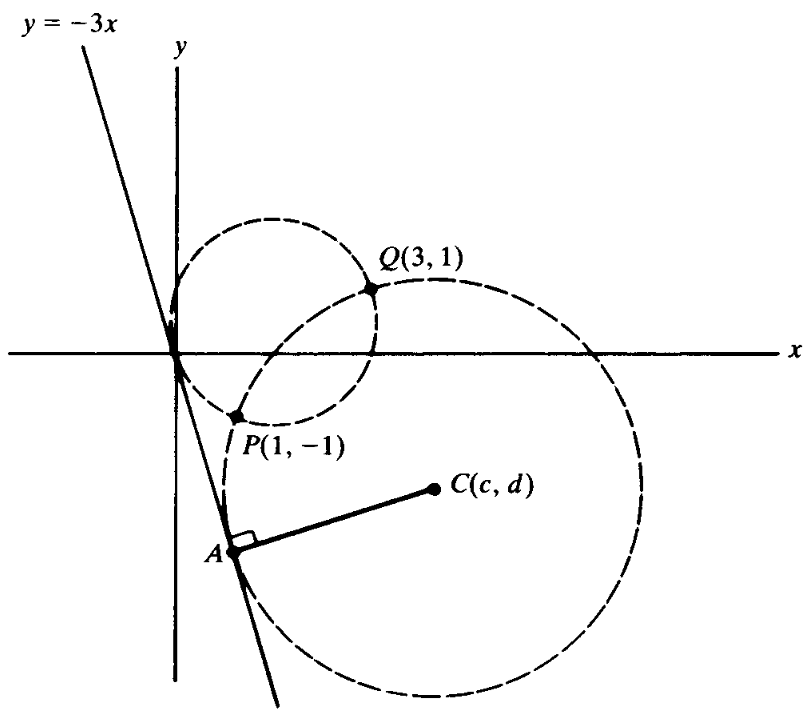


Fig. 4-4

In addition, $\overline{CP} = \overline{CA}$, and by the formula of Problem 8 in Chapter 3, $\overline{CA} = \dfrac{3c+d}{\sqrt{10}}$. Setting $\overline{CP}^2 = \overline{CA}^2$ thus yields $(c-1)^2 + (d+1)^2 = \dfrac{(3c+d)^2}{10}$. Substituting (*1*) in the right-hand side and multiplying by 10 then yields

$$10[(c-1)^2 + (d+1)^2] = (2c+2)^2 \qquad \text{from which} \qquad 3c^2 + 5d^2 - 14c + 10d + 8 = 0$$

By (*1*), we can replace d by $2 - c$, obtaining

$$2c^2 - 11c + 12 = 0 \qquad \text{or} \qquad (2c-3)(c-4) = 0$$

Hence, $c = \frac{3}{2}$ or $c = 4$. Then (*1*) gives us the two solutions $c = \frac{3}{2}$, $d = \frac{1}{2}$ and $c = 4$, $d = -2$. Since the radius $CA = \dfrac{3c+d}{\sqrt{10}}$, these solutions produce radii of $\dfrac{10/2}{\sqrt{10}} = \dfrac{\sqrt{10}}{2}$ and $\dfrac{10}{\sqrt{10}} = \sqrt{10}$. Thus, there are two such circles, and their standard equations are

$$(x - \tfrac{3}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{5}{2} \qquad \text{and} \qquad (x-4)^2 + (y+2)^2 = 10$$

Supplementary Problems

6. Find the standard equations of the circles satisfying the following conditions:
(*a*) center at $(3, 5)$ and radius 2 (*b*) center at $(4, -1)$ and radius 1
(*c*) center at $(5, 0)$ and radius $\sqrt{3}$ (*d*) center at $(-2, -2)$ and radius $5\sqrt{2}$
(*e*) center at $(-2, 3)$ and passing through $(3, -2)$
(*f*) center at $(6, 1)$ and passing through the origin

Ans. (*a*) $(x-3)^2 + (y-5)^2 = 4$; (*b*) $(x-4)^2 + (y+1)^2 = 1$; (*c*) $(x-5)^2 + y^2 = 3$; (*d*) $(x+2)^2 + (y+2)^2 = 50$; (*e*) $(x+2)^2 + (y-3)^2 = 50$; (*f*) $(x-6)^2 + (y-1)^2 = 37$

7. Identify the graphs of the following equations:
(*a*) $x^2 + y^2 + 16x - 12y + 10 = 0$ (*b*) $x^2 + y^2 - 4x + 5y + 10 = 0$ (*c*) $x^2 + y^2 + x - y = 0$
(*d*) $4x^2 + 4y^2 + 8y - 3 = 0$ (*e*) $x^2 + y^2 - x - 2y + 3 = 0$ (*f*) $x^2 + y^2 + \sqrt{2}x - 2 = 0$

Ans. (*a*) circle, center at $(-8, 6)$, radius $3\sqrt{10}$; (*b*) circle, center at $(2, -\frac{5}{2})$, radius $\frac{1}{2}$; (*c*) circle, center at $(-\frac{1}{2}, \frac{1}{2})$, radius $\sqrt{2}/2$; (*d*) circle, center at $(0, -1)$, radius $\frac{7}{2}$; (*e*) empty graph; (*f*) circle, center at $(-\sqrt{2}/2, 0)$, radius $\sqrt{5/2}$

8. Find the standard equations of the circles through (*a*) $(-2, 1)$, $(1, 4)$, and $(-3, 2)$; (*b*) $(0, 1)$, $(2, 3)$, and $(1, 1+\sqrt{3})$; (*c*) $(6, 1)$, $(2, -5)$, and $(1, -4)$; (*d*) $(2, 3)$, $(-6, -3)$, and $(1, 4)$.

Ans. (*a*) $(x+1)^2 + (y-3)^2 = 5$; (*b*) $(x-2)^2 + (y-1)^2 = 4$; (*c*) $(x-4)^2 + (y+2)^2 = 13$; (*d*) $(x+2)^2 + y^2 = 25$

9. For what values of k does the circle $(x+2k)^2 + (y-3k)^2 = 10$ pass through the point $(1, 0)$?

Ans. $k = \frac{9}{13}$ or $k = -1$

10. Find the standard equations of the circles of radius 2 that are tangent to both the lines $x = 1$ and $y = 3$.

Ans. $(x+1)^2 + (y-1)^2 = 4$; $(x+1)^2 + (y-5)^2 = 4$; $(x-3)^2 + (y-1)^2 = 4$; $(x-3)^2 + (y-5)^2 = 4$

11. Find the value of k so that $x^2 + y^2 + 4x - 6y + k = 0$ is the equation of a circle of radius 5.

Ans. $k = -12$

12. Find the standard equation of the circle having as a diameter the segment joining $(2, -3)$ and $(6. 5)$.

Ans. $(x-4)^2+(y-1)^2=20$

13. Find the standard equation of every circle that passes through the origin, has radius 5, and is such that the y coordinate of its center is -4.

Ans. $(x-3)^2+(y+4)^2=25$ or $(x+3)^2+(y+4)^2=25$

14. Find the standard equation of the circle that passes through the points $(8, -5)$ and $(-1, 4)$ and has its center on the line $2x+3y=3$.

Ans. $(x-3)^2+(y+1)^2=41$

15. Find the standard equation of the circle with center $(3, 5)$ that is tangent to the line $12x-5y+2=0$.

Ans. $(x-3)^2+(y-5)^2=1$

16. Find the standard equation of the circle that passes through the point $(1, 3+\sqrt{2})$ and is tangent to the line $x+y=2$ at $(2, 0)$.

Ans. $(x-5)^2+(y-3)^2=18$

17. Prove analytically that an angle inscribed in a semicircle is a right angle. (See Fig. 4-5.)

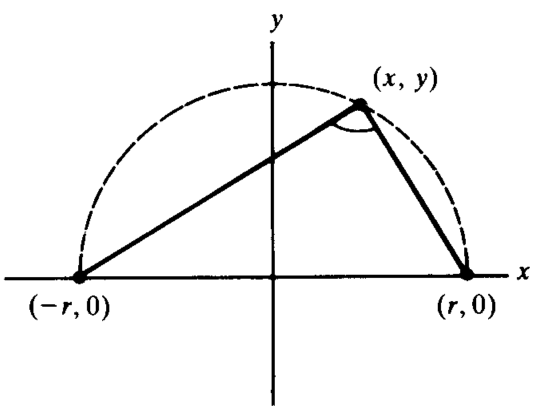


Fig. 4-5

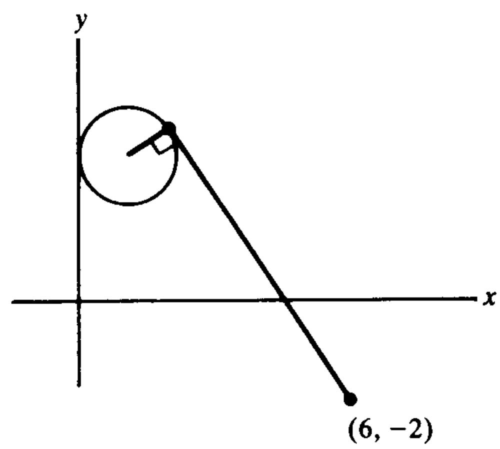


Fig. 4-6

18. Find the length of a tangent from $(6, -2)$ to the circle $(x-1)^2+(y-3)^2=1$. (See Fig. 4-6.)

Ans. 7

19. Find the standard equations of the circles that pass through $(2, 3)$ and are tangent to both the lines $3x-4y=-1$ and $4x+3y=7$.

Ans. $(x-2)^2+(y-8)^2=25$ and $(x-\frac{6}{5})^2+(y-\frac{12}{5})^2=1$

20. Find the standard equations of the circles that have their centers on the line $4x+3y=8$ and are tangent to both the lines $x+y=-2$ and $7x-y=-6$.

Ans. $(x-2)^2+y^2=2$ and $(x+4)^2+(y-8)^2=18$

21. Find the standard equation of the circle that is concentric with the circle $x^2+y^2-2x-8y+1=0$ and is tangent to the line $2x-y=3$.

Ans. $(x-1)^2+(y-4)^2=5$

22. Find the standard equations of the circles that have radius 10 and are tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Ans. $(x-9)^2 + (y-12)^2 = 100$ and $(x+3)^2 + (y+4)^2 = 100$

23. Find the longest and shortest distances from the point $(7, 12)$ to the circle $x^2 + y^2 + 2x + 6y - 15 = 0$.

Ans. 22 and 12

24. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two intersecting circles determined by the equations $x^2 + y^2 + A_1x + B_1y + C_1 = 0$ and $x^2 + y^2 + A_2x + B_2y + C_2 = 0$. For any number $k \neq -1$, show that

$$x^2 + y^2 + A_1x + B_1y + C_1 + k(x^2 + y^2 + A_2x + B_2y + C_2) = 0$$

is the equation of a circle through the intersection points of $\mathscr{C}_1$ and $\mathscr{C}_2$. Show, conversely, that every such circle may be represented by such an equation for a suitable k.

25. Find the standard equation of the circle passing through the point $(-3, 1)$ and containing the points of intersection of the circles $x^2 + y^2 + 5x = 1$ and $x^2 + y^2 + y = 7$.

Ans. $(x + \frac{5}{9})^2 + (y + \frac{7}{18})^2 = \frac{569}{100}$

26. Find the standard equations of the circles that have centers on the line $5x - 2y = -21$ and are tangent to both coordinate axes.

Ans. $(x+7)^2 + (y+7)^2 = 49$ and $(x+3)^2 + (y-3)^2 = 9$

27. (*a*) If two circles $x^2 + y^2 + A_1x + B_1y + C_1 = 0$ and $x^2 + y^2 + A_2x + B_2y + C_2 = 0$ intersect at two points, find an equation of the line through their points of intersection.
(*b*) Prove that if two circles intersect at two points, then the line through their points of intersection is perpendicular to the line through their centers.

Ans. (*a*) $(A_1 - A_2)x + (B_1 - B_2)y + (C_1 - C_2) = 0$

28. Find the points of intersection of the circles $x^2 + y^2 + 8y - 64 = 0$ and $x^2 + y^2 - 6x - 16 = 0$.

Ans. $(8, 0)$ and $(\frac{24}{15}, \frac{24}{5})$

29. Find the equations of the lines through $(4, 10)$ and tangent to the circle $x^2 + y^2 - 4y - 36 = 0$.

Ans. $y = -3x + 22$ and $x - 3y + 26 = 0$

Chapter 5

Equations and Their Graphs

THE GRAPH OF AN EQUATION involving x and y as its only variables consists of all points (x, y) satisfying the equation.

EXAMPLE 1: (*a*) What is the graph of the equation $2x - y = 3$?

The equation is equivalent to $y = 2x - 3$, which we know is the slope-intercept equation of the line with slope 2 and y intercept -3.

(*b*) What is the graph of the equation $x^2 + y^2 - 2x + 4y - 4 = 0$?

Completing the square shows that the given equation is equivalent to the equation $(x-1)^2 + (y+2)^2 = 9$. Hence, its graph is the circle with center $(1, -2)$ and radius 3.

PARABOLAS. Consider the equation $y = x^2$. If we substitute a few values for x and calculate the associated values of y, we obtain the results tabulated in Fig. 5-1. We can plot the corresponding points, as shown in the figure. These points suggest the heavy curve, which belongs to a family of curves called *parabolas*. In particular, the graphs of equations of the form $y = cx^2$, where c is a nonzero constant, are parabolas, as are any other curves obtained from them by translations and rotations.

x	y
3	9
2	4
1	1
0	0
-1	1
-2	4
-3	9

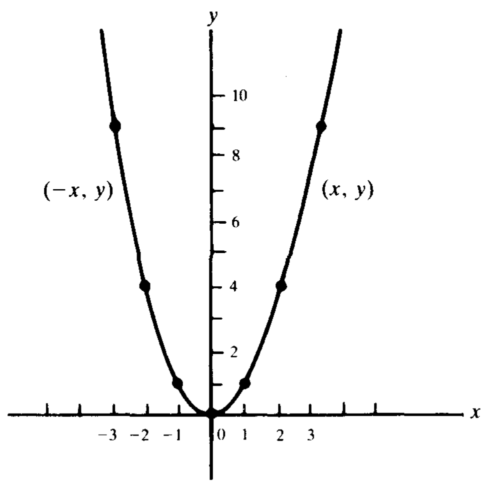


Fig. 5-1

In Fig. 5-1, we note that the graph of $y = x^2$ contains the origin $(0, 0)$ but all its other points lie above the x axis, since x^2 is positive except when $x = 0$. When x is positive and increasing, y increases without bound. Hence, in the first quadrant, the graph moves up without bound as it moves right. Since $(-x)^2 = x^2$, it follows that, if any point (x, y) lies on the graph in the first quadrant, then the point $(-x, y)$ also lies on the graph in the second quadrant. Thus, the graph is symmetric with respect to the y axis. The y axis is called the *axis of symmetry* of this parabola.

ELLIPSES. To construct the graph of the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, we again compute a few values and plot the corresponding points, as shown in Fig. 5-2. The graph suggested by these points is also

x	y
3	0
2	$\pm\frac{2}{3}\sqrt{5} \approx \pm 1.5$
1	$\pm\frac{4}{3}\sqrt{2} \approx \pm 1.9$
0	± 2
-1	$\pm\frac{4}{3}\sqrt{2}$
-2	$\pm\frac{2}{3}\sqrt{5}$
-3	0

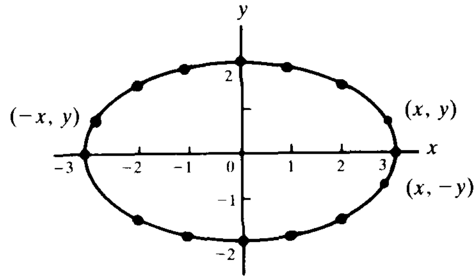


Fig. 5-2

drawn in the figure; it is a member of a family of curves called *ellipses*. In particular, the graph of an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse, as is any curve obtained from it by translation or rotation.

Note that, in contrast to parabolas, ellipses are bounded. In fact, if (x, y) is on the graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$, then $\frac{x^2}{9} \le \frac{x^2}{9} + \frac{y^2}{4} = 1$, and, therefore, $x^2 \le 9$. Hence, $-3 \le x \le 3$. So, the graph lies between the vertical lines $x = -3$ and $x = 3$. Its rightmost point is $(3, 0)$, and its leftmost point is $(-3, 0)$. A similar argument shows that the graph lies between the horizontal lines $y = -2$ and $y = 2$, and that its lowest point is $(0, -2)$ and its highest point is $(0, 2)$. In the first quadrant, as x increases from 0 to 3, y decreases from 2 to 0. If (x, y) is any point on the graph, then $(-x, y)$ also is on the graph. Hence, the graph is symmetric with respect to the y axis. Similarly, if (x, y) is on the graph, so is $(x, -y)$, and therefore the graph is symmetric with respect to the x axis.

When $a = b$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the circle with the equation $x^2 + y^2 = a^2$, that is, a circle with center at the origin and radius a. Thus, circles are special cases of ellipses.

HYPERBOLAS. Consider the graph of the equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Some of the points on this graph are tabulated and plotted in Fig. 5-3. These points suggest the curve shown in the figure, which is a member of a family of curves called *hyperbolas*. The graphs of equations of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are hyperbolas, as are any curves obtained from them by translations and rotations.

Let us look at the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ in more detail. Since $\frac{x^2}{9} = 1 + \frac{y^2}{4} \ge 1$, it follows that $x^2 \ge 9$, and therefore, $|x| \ge 3$. Hence, there are no points on the graph between the vertical lines $x = -3$ and $x = 3$. If (x, y) is on the graph, so is $(-x, y)$; thus, the graph is symmetric with respect to the y axis. Similarly, the graph is symmetric with respect to the x axis. In the first quadrant, as x increases, y increases without bound.

Note the dashed lines in Fig. 5-3; they are the lines $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$, and they called the *asymptotes* of the hyperbola: Points on the hyperbola get closer and closer to these asymptotes as they recede from the origin. In general, *the asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are the lines* $y = \frac{b}{a}x$ *and* $y = -\frac{b}{a}x$.

CONIC SECTIONS. Parabolas, ellipses, and hyperbolas together make up a class of curves called *conic sections*. They can be defined geometrically as the intersections of planes with the surface of a right circular cone, as shown in Fig. 5-4.

x	y
± 3	0
± 4	$\pm \frac{2}{3}\sqrt{7} \approx \pm 1.76$
± 5	$\pm \frac{8}{3} \approx \pm 2.67$
± 6	$\pm 2\sqrt{3} \approx \pm 3.46$

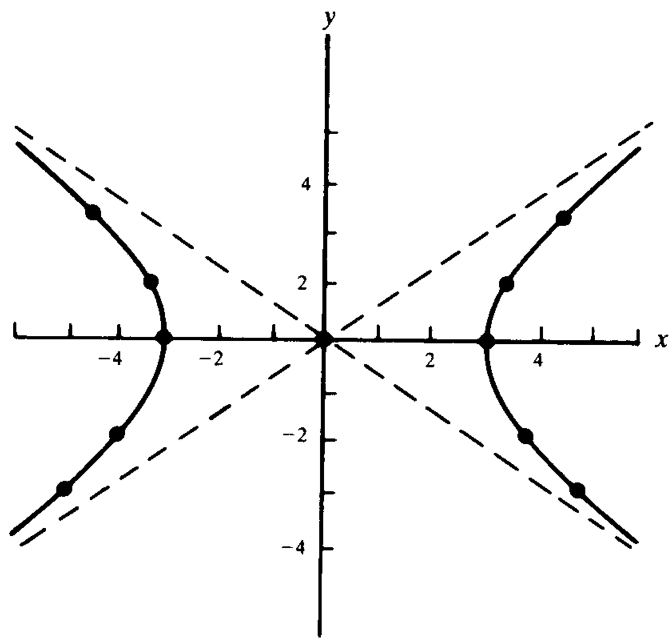


Fig. 5-3

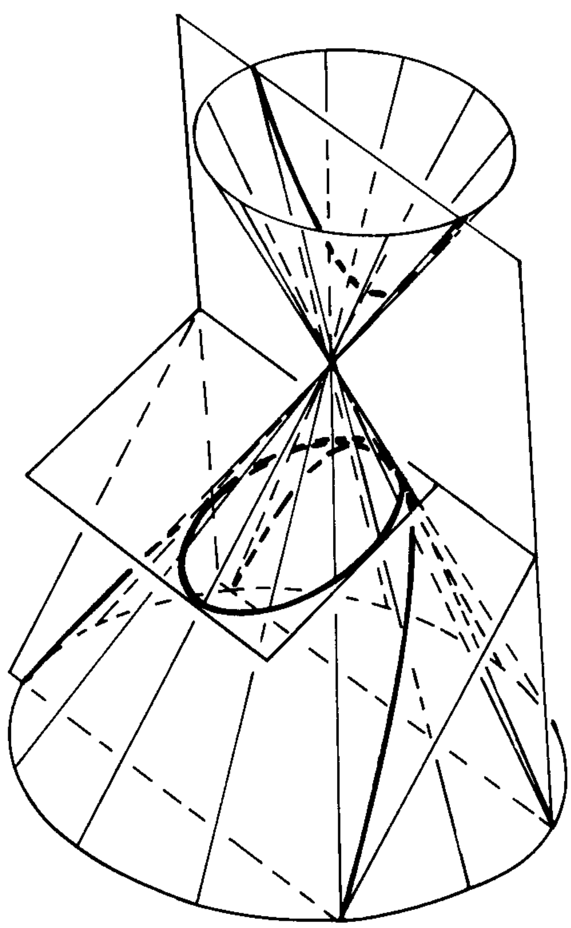


Fig. 5-4

Solved Problems

1. Sketch the graph of the *cubic curve* $y = x^3$.

The graph passes through the origin $(0, 0)$. Also, for any point (x, y) on the graph, x and y have the same sign; hence, the graph lies in the first and third quadrants. In the first quadrant, as x increases, y increases without bound. Moreover, if (x, y) lies on the graph, then $(-x, -y)$ also lies on the graph. Since the origin is the midpoint of the segment connecting the points (x, y) and $(-x, -y)$, the graph is symmetric with respect to the origin. Some points on the graph are tabulated and shown in Fig. 5-5; these points suggest the heavy curve in the figure.

x	y
0	0
1/2	1/8
1	1
3/2	27/8
2	8
−1/2	−1/8
−1	−1
−3/2	−27/8
−2	−8

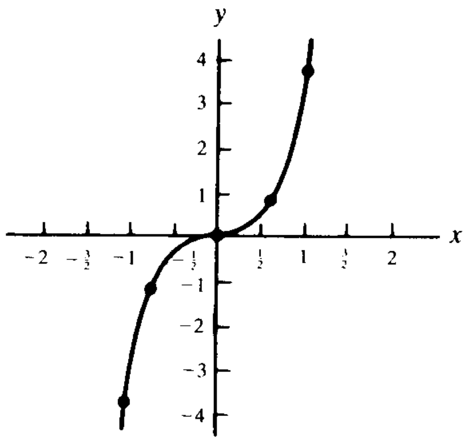


Fig. 5-5

2. Sketch the graph of the equation $y = -x^2$.

If (x, y) is on the graph of the parabola $y = x^2$ (Fig. 5-1), then $(x, -y)$ is on the graph of $y = -x^2$, and vice versa. Hence, the graph of $y = -x^2$ is the reflection in the x axis of the graph of $y = x^2$. The result is the parabola in Fig. 5-6.

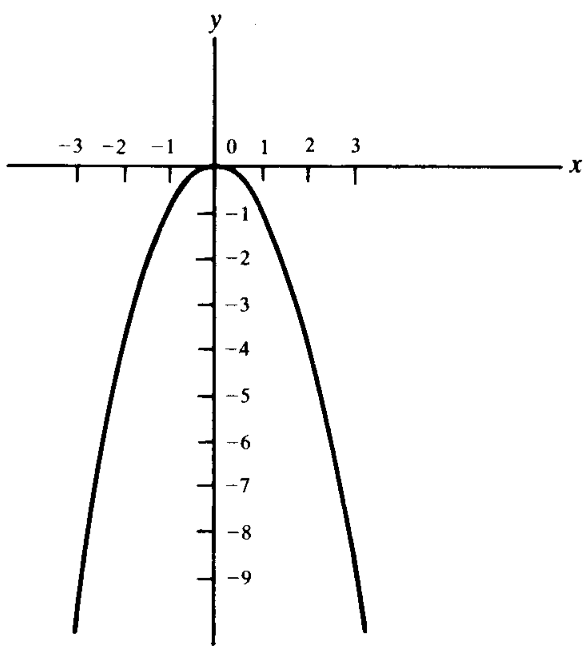


Fig. 5-6

3. Sketch the graph of $x = y^2$.

This graph is obtained from the parabola $y = x^2$ by exchanging the roles of x and y. The resulting curve is a parabola with the x axis as its axis of symmetry and its "nose" at the origin (see Fig. 5-7). A point (x, y) is on the graph of $x = y^2$ if and only if (y, x) is on the graph of $y = x^2$. Since the segment connecting the points (x, y) and (y, x) is perpendicular to the diagonal line $y = x$ (why?), and the midpoint $\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ of that segment is on the line $y = x$ (see Fig. 5-8), the parabola $x = y^2$ is obtained from the parabola $y = x^2$ by reflection in the line $y = x$.

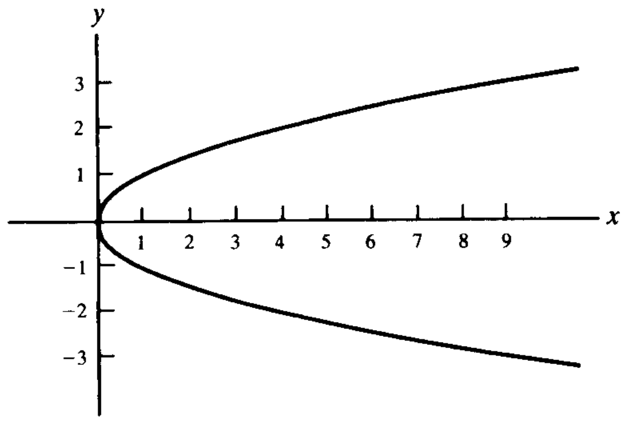


Fig. 5-7

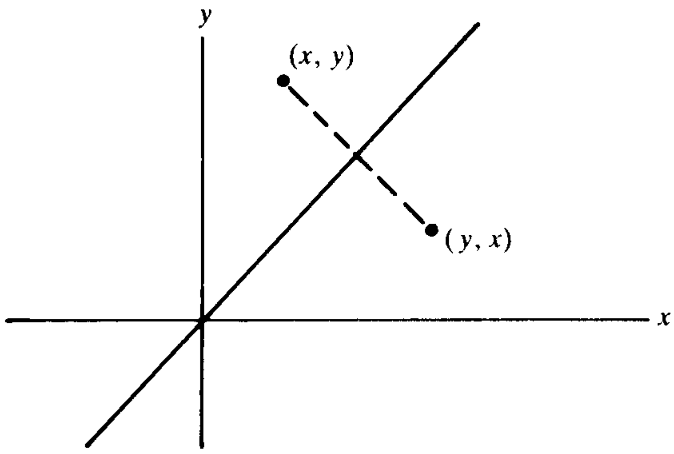


Fig. 5-8

4. Let $\mathscr{L}$ be a line, and let F be a point not on $\mathscr{L}$. Show that the set of all points equidistant from F and $\mathscr{L}$ is a parabola.

Construct a coordinate system such that F lies on the positive y axis, and the x axis is parallel to $\mathscr{L}$ and halfway between F and $\mathscr{L}$. (See Fig. 5-9.) Let $2p$ be the distance between F and $\mathscr{L}$. Then $\mathscr{L}$ has the equation $y = -p$, and the coordinates of F are $(0, p)$.

Consider an arbitrary point $P(x, y)$. Its distance from $\mathscr{L}$ is $|y + p|$, and its distance from F is $\sqrt{x^2 + (y - p)^2}$. Thus, for the point to be equidistant from F and $\mathscr{L}$ we must have $|y + p| = \sqrt{x^2 + (y - p)^2}$. Squaring yields $(y + p)^2 = x^2 + (y - p)^2$, from which we find that $4py = x^2$. This is the equation of a parabola with the y axis as its axis of symmetry. The point F is called the *focus* of the parabola, and the line $\mathscr{L}$ is called its *directrix*. The chord AB through the focus and parallel to $\mathscr{L}$ is called the *latus rectum*. The "nose" of the parabola at $(0, 0)$ is called its *vertex*.

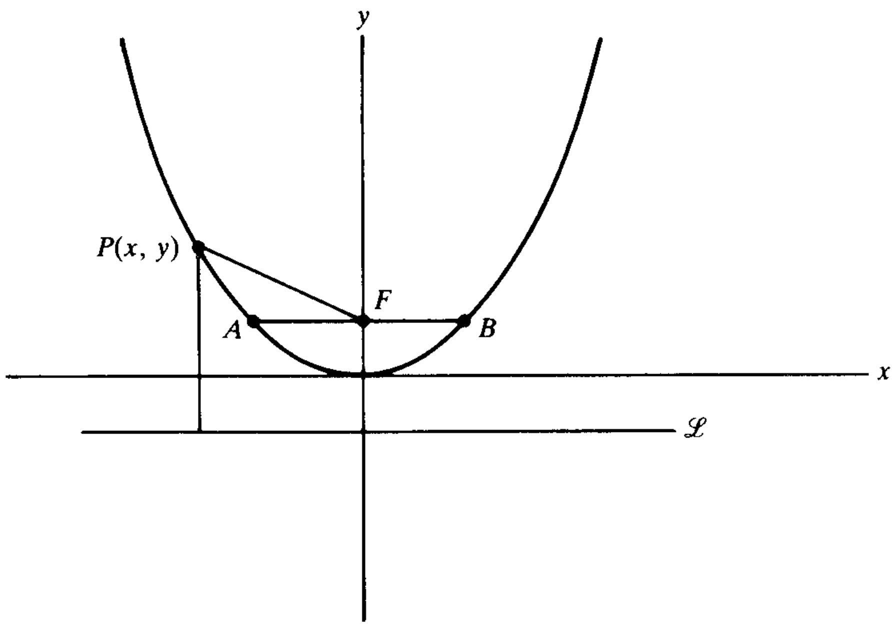


Fig. 5-9

5. Find the length of the latus rectum of a parabola $4py = x^2$.

The y coordinate of the endpoints A and B of the latus rectum (see Fig. 5-9) is p. Hence, at these points, $4p^2 = x^2$ and, therefore, $x = \pm 2p$. Thus, the length AB of the latus rectum is $4p$.

6. Find the focus, directrix, and the length of the latus rectum of the parabola $y = \frac{1}{2}x^2$, and draw its graph.

The equation of the parabola can be written as $2y = x^2$. Hence, $4p = 2$ and $p = \frac{1}{2}$. Therefore, the focus is at $(0, \frac{1}{2})$, the equation of the directix is $y = -\frac{1}{2}$, and the length of the latus rectum is 2. The graph is shown in Fig. 5-10.

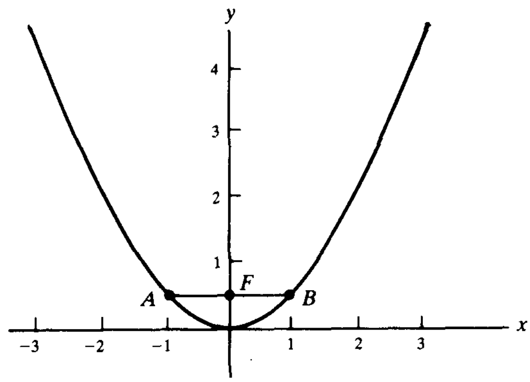


Fig. 5-10

7. Let F and F' be two distinct points at a distance $2c$ from each other. Show that the set of all points $P(x, y)$ such that $\overline{PF} + \overline{PF'} = 2a$, $a > c$, is an ellipse.

Construct a coordinate system such that the x axis passes through F and F', the origin is the midpoint of the segment FF', and F lies on the positive x axis. Then the coordinates of F and F' are $(c, 0)$ and $(-c, 0)$. (See Fig. 5-11.) Thus, the condition $\overline{PF} + \overline{PF'} = 2a$ is equivalent to $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$. After rearranging and squaring twice (to eliminate the square roots) and performing indicated operations, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \tag{1}$$

Since $a > c$, $a^2 - c^2 > 0$. Let $b = \sqrt{a^2 - c^2}$. Then (*1*) becomes $b^2x^2 + a^2y^2 = a^2b^2$, which we may rewrite as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of an ellipse.

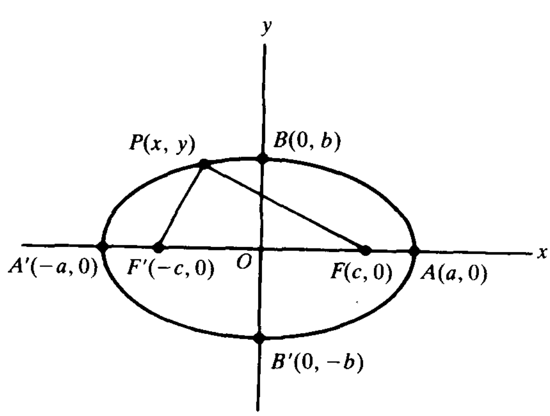


Fig. 5-11

When $y = 0$, $x^2 = a^2$; hence, the ellipse intersects the x axis at the points $A'(-a, 0)$, and $A(a, 0)$, called the *vertices* of the ellipse (Fig. 5-11). The segment $A'A$ is called the *major axis*; the segment OA is called the *semimajor axis* and has length a. The origin is the *center* of the ellipse. F and F' are called the *foci* (each is a *focus*). When $x = 0$, $y^2 = b^2$. Hence, the ellipse intersects the y axis at the points $B'(0, -b)$ and $B(0, b)$. The segment $B'B$ is called the *minor axis*; the segment OB is called the *semiminor axis* and has length b. Note that $b = \sqrt{a^2 - c^2} < \sqrt{a^2} = a$. Hence, the semiminor axis is smaller than the semimajor axis. The basic relation among a, b, and c is $a^2 = b^2 + c^2$.

The *eccentricity* of an ellipse is defined to be $e = c/a$. Note that $0 < e < 1$. Moreover, $e = \sqrt{a^2 - b^2}/a = \sqrt{1 - (b/a)^2}$. Hence, when e is very small, b/a is very close to 1, the minor axis is close in size to the major axis, and the ellipse is close to being a circle. On the other hand, when e is close to 1, b/a is close to zero, the minor axis is very small in comparison with the major axis, and the ellipse is very "flat."

8. Identify the graph of the equation $9x^2 + 16y^2 = 144$.

The given equation is equivalent to $x^2/16 + y^2/9 = 1$. Hence, the graph is an ellipse with semimajor axis of length $a = 4$ and semiminor axis of length $b = 3$. (See Fig. 5-12.) The vertices are $(-4, 0)$ and $(4, 0)$. Since $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$, the eccentricity e is $c/a = \sqrt{7}/4 \approx 0.6614$.

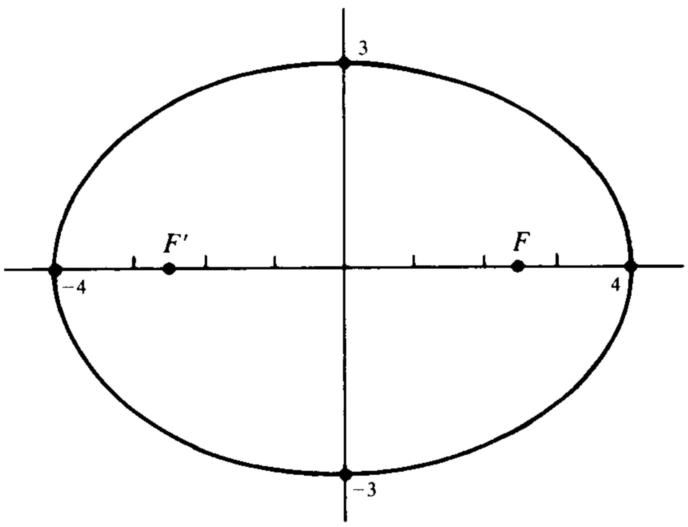


Fig. 5-12

9. Identify the graph of the equation $25x^2 + 4y^2 = 100$.

The given equation is equivalent to $x^2/4 + y^2/25 = 1$, an ellipse. Since the denominator under y^2 is larger than the denominator under x^2, the graph is an ellipse with the major axis on the y axis and the minor axis on the x axis (see Fig. 5-13). The vertices are at $(0, -5)$ and $(0, 5)$. Since $c = \sqrt{a^2 - b^2} = \sqrt{21}$, the eccentricity is $\sqrt{21}/5 \approx 0.9165$.

10. Let F and F' be distinct points at a distance of $2c$ from each other. Find the set of all points $P(x, y)$ such that $|\overline{PF} - \overline{PF'}| = 2a$, for $a < c$.

Choose a coordinate system such that the x axis passes through F and F', with the origin as the midpoint of the segment FF' and with F on the positive x axis (see Fig. 5-14). The coordinates of F and F' are $(c, 0)$ and $(-c, 0)$. Hence, the given condition is equivalent to $\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = \pm 2a$. After manipulations required to eliminate the square roots, this yields

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \qquad (1)$$

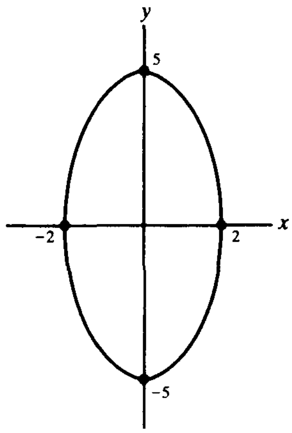


Fig. 5-13

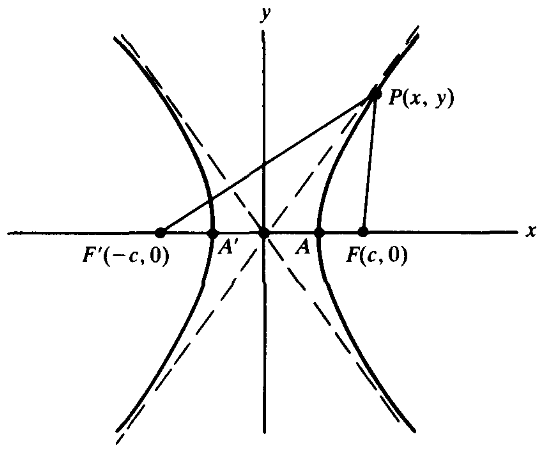


Fig. 5-14

Since $c > a$, $c^2 - a^2 > 0$. Let $b = \sqrt{c^2 - a^2}$. (Notice that $a^2 + b^2 = c^2$.) Then (*1*) becomes $b^2x^2 - a^2y^2 = a^2b^2$, which we rewrite as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the equation of a hyperbola.

When $y = 0$, $x = \pm a$. Hence, the hyperbola intersects the x axis at the points $A'(-a, 0)$ and $A(a, 0)$, which are called the *vertices* of the hyperbola. The asymptotes are $y = \pm \frac{b}{a} x$. The segment $A'A$ is called the *transverse axis*. The segment connecting the points $(0, -b)$ and $(0, b)$ is called the *conjugate axis*. The *center* of the hyperbola is the origin. The points F and F' are called the *foci*. The *eccentricity* is defined to be $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \left(\frac{b}{a}\right)^2}$. Since $c > a$, $e > 1$. When e is close to 1, b is very small relative to a, and the hyperbola has a very pointed "nose"; when e is very large, b is very large relative to a, and the hyperbola is very "flat."

11. Identify the graph of the equation $25x^2 - 16y^2 = 400$.

The given equation is equivalent to $x^2/16 - y^2/25 = 1$. This is the equation of a hyperbola with the x axis as its transverse axis, vertices $(-4, 0)$ and $(4, 0)$, and asymptotes $y = \pm \frac{5}{4}x$. (See Fig. 5-15.)

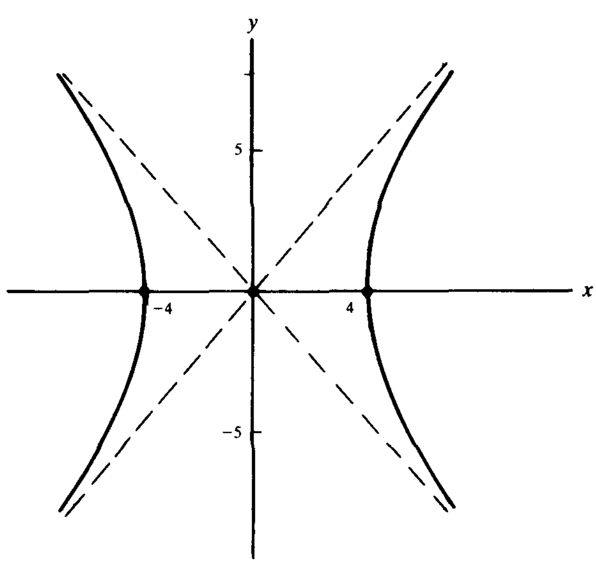


Fig. 5-15

12. Identify the graph of the equation $y^2 - 4x^2 = 4$.

The given equation is equivalent to $\frac{y^2}{4} - \frac{x^2}{1} = 1$. This is the equation of a hyperbola, with the roles of x and y interchanged. Thus, the transverse axis is the y axis, the conjugate axis is the x axis, and the vertices are $(0, -2)$ and $(0, 2)$. The asymptotes are $x = \pm \frac{1}{2}y$ or, equivalently, $y = \pm 2x$. (See Fig. 5-16.)

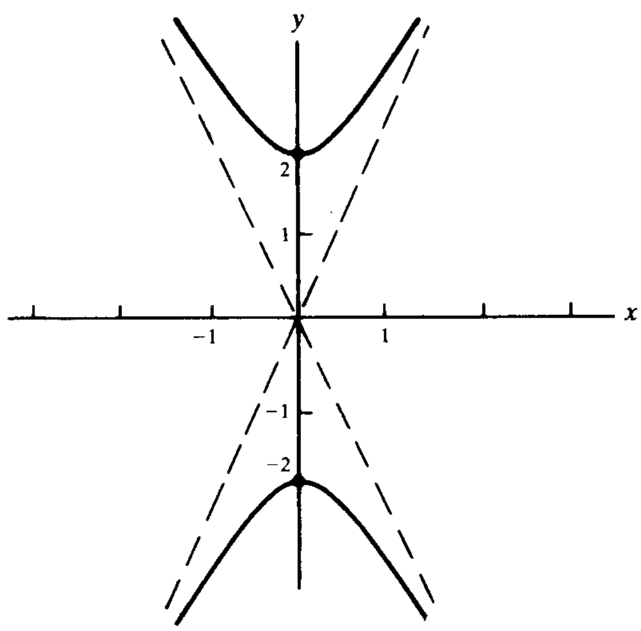


Fig. 5-16

13. Identify the graph of the equation $y = (x - 1)^2$.

A point (u, v) is on the graph of $y = (x - 1)^2$ if and only if the point $(u - 1, v)$ is on the graph of $y = x^2$. Hence, the desired graph is obtained from the parabola $y = x^2$ by moving each point of the latter one unit to the right. (See Fig. 5-17.)

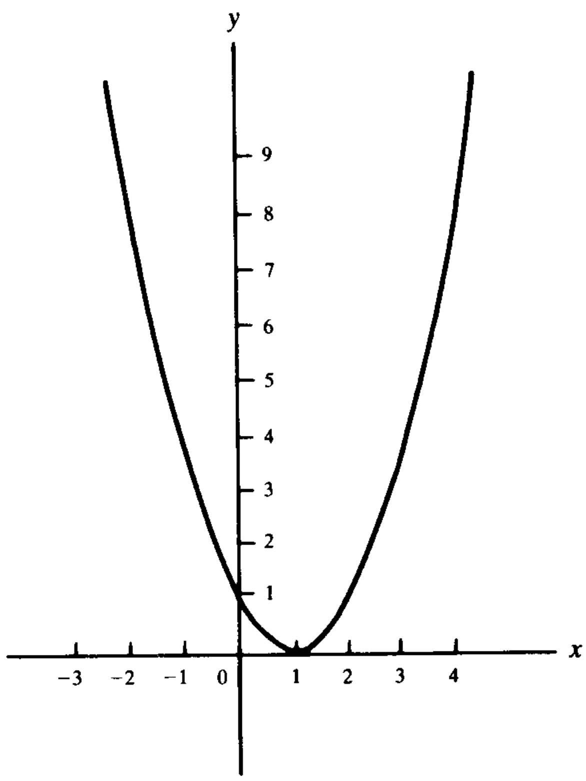


Fig. 5-17

14. Identify the graph of the equation $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$.

A point (u, v) is on the graph if and only if the point $(u-1, v-2)$ is on the graph of the equation $x^2/4 + y^2/9 = 1$. Hence, the desired graph is obtained by moving the ellipse $x^2/4 + y^2/9 = 1$ one unit to the right and two units upward. (See Fig. 5-18.) The center of the ellipse is at $(1, 2)$, the major axis is along the line $x = 1$, and the minor axis is along the line $y = 2$.

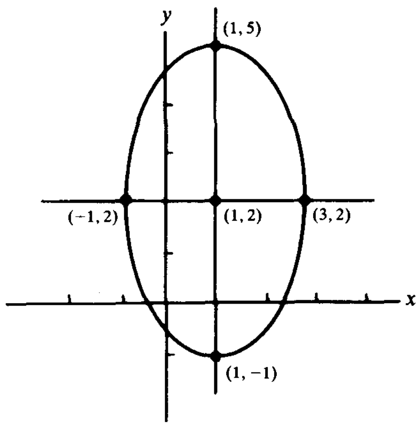


Fig. 5-18

15. How is the graph of an equation $F(x-a, y-b) = 0$ related to the graph of the equation $F(x, y) = 0$?

A point (u, v) is on the graph of $F(x-a, y-b) = 0$ if and only if the point $(u-a, v-b)$ is on the graph of $F(x, y) = 0$. Hence, the graph of $F(x-a, y-b) = 0$ is obtained by moving each point of the graph of $F(x, y) = 0$ by a units to the right and b units upward. (If a is negative, we move the point $|a|$ units to the left. If b is negative, we move the point $|b|$ units downward.) Such a motion is called a *translation*.

16. Identify the graph of the equation $y = x^2 - 2x$.

Completing the square in x, we obtain $y + 1 = (x-1)^2$. Based on the results of Problem 15, the graph is obtained by a translation of the parabola $y = x^2$ so that the new vertex is $(1, -1)$. [Notice that $y + 1$ is $y - (-1)$.] It is shown in Fig. 5-19.

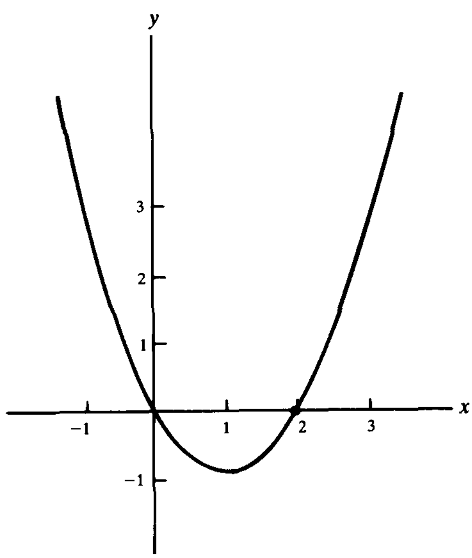


Fig. 5-19

17. Identify the graph of $4x^2 - 9y^2 - 16x + 18y - 29 = 0$.

Factoring yields $4(x^2 - 4x) - 9(y^2 - 2y) - 29 = 0$, and then completing the square in x and y produces $4(x-2)^2 - 9(y-1)^2 = 36$. Dividing by 36 then yields $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 1$. By the results of Problem 15, the graph of this equation is obtained by translating the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ two units to the right and one unit upward, so that the new center of symmetry of the hyperbola is $(2, 1)$. (See Fig. 5-20.)

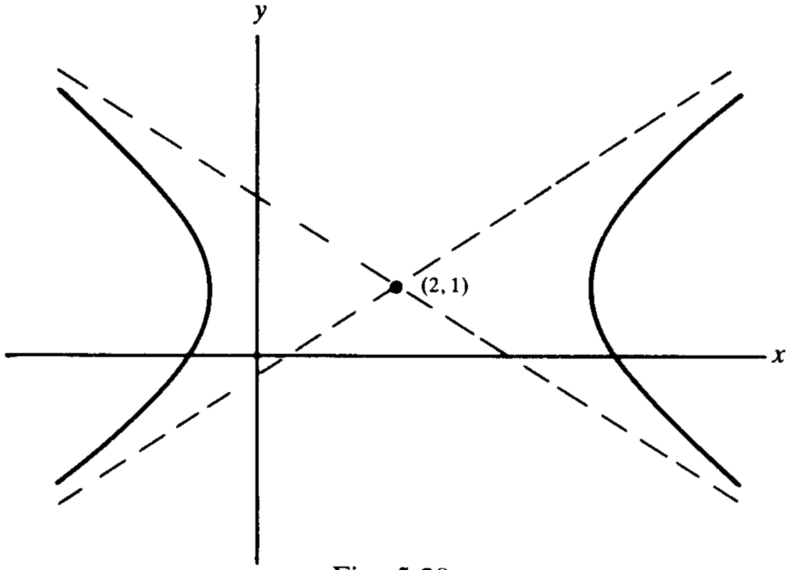


Fig. 5-20

18. Draw the graph of the equation $xy = 1$.

Some points of the graph are tabulated and plotted in Fig. 5-21. The curve suggested by these points is shown dashed as well. It can be demonstrated that this curve is a hyperbola with the line $y = x$ as transverse axis, the line $y = -x$ as converse axis, vertices $(-1, -1)$ and $(1, 1)$, and the x axis and y axis as asymptotes. Similarly, the graph of any equation $xy = d$, where d is a positive constant, is a hyperbola with $y = x$ as transverse axis and $y = -x$ as converse axis, and with the coordinate axes as asymptotes. Such hyperbolas are called *equilateral hyperbolas*. They can be shown to be rotations of hyperbolas of the form $x^2/a^2 - y^2/a^2 = 1$.

x	y
3	1/3
2	1/2
1	1
1/2	2
1/3	3
1/4	4
−1/4	−4
−1/3	−3
−1/2	−2
−1	−1
−2	−1/2
−3	−1/3

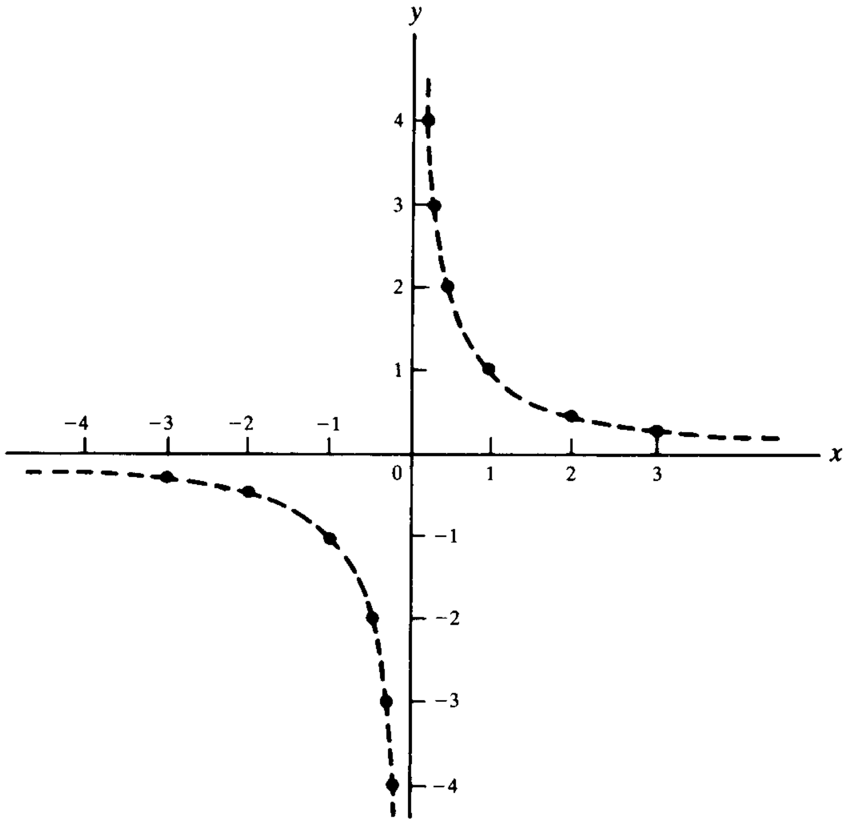


Fig. 5-21

Supplementary Problems

19. On the same sheet of paper, draw the graphs of the following parabolas: (*a*) $y = 2x^2$; (*b*) $y = 3x^2$; (*c*) $y = 4x^2$; (*d*) $y = \frac{1}{2}x^2$; (*e*) $y = \frac{1}{3}x^2$.

20. On the same sheet of paper, draw the graphs of the following parabolas, and indicate points of intersection: (*a*) $y = x^2$; (*b*) $y = -x^2$; (*c*) $x = y^2$; (*d*) $x = -y^2$.

21. Draw the graphs of the following equations:
(*a*) $y = x^3 - 1$ (*b*) $y = (x-2)^3$ (*c*) $y = (x+1)^3 - 2$
(*d*) $y = -x^3$ (*e*) $y = -(x-1)^3$ (*f*) $y = -(x-1)^3 + 2$

22. Identify and draw the graphs of the following equations:
(*a*) $y^2 - x^2 = 1$ (*b*) $25x^2 + 36y^2 = 900$ (*c*) $2x^2 - y^2 = 4$ (*d*) $xy = 4$
(*e*) $4x^2 + 4y^2 = 1$ (*f*) $8x = y^2$ (*g*) $10y = x^2$ (*h*) $4x^2 + 9y^2 = 16$
(*i*) $xy = -1$ (*j*) $3y^2 - x^2 = 9$

Ans. (*a*) hyperbola, y axis as transverse axis, vertices $(0, \pm 1)$, asymptotes $y = \pm x$; (*b*) ellipse, vertices $(\pm 6, 0)$ foci $(\pm\sqrt{11}, 0)$; (*c*) hyperbola, x axis as transverse axis, vertices $(\pm\sqrt{2}, 0)$, asymptotes $y = \pm\sqrt{2}x$; (*d*) hyperbola, $y = x$ as transverse axis, vertices $(2, 2)$ and $(-2, -2)$, x and y axes as asymptotes; (*e*) circle, center $(0, 0)$, radius $\frac{1}{2}$; (*f*) parabola, vertex $(0, 0)$, focus $(2, 0)$, directrix $x = -2$; (*g*) parabola, vertex $(0, 0)$, focus $(0, \frac{5}{2})$, directrix $y = -\frac{5}{2}$; (*h*) ellipse, vertices $(\pm 2, 0)$, foci $(\pm\frac{2}{3}\sqrt{5}, 0)$; (*i*) hyperbola, $y = -x$ as transverse axis, vertices $(-1, 1)$ and $(1, -1)$, x and y axes as asymptotes; (*j*) hyperbola, y axis as transverse axis, vertices $(0, \pm\sqrt{3})$, asymptotes $y = \pm\sqrt{3}x/3$

23. Identify and draw the graphs of the following equations:
(*a*) $4x^2 - 3y^2 + 8x + 12y - 4 = 0$ (*b*) $5x^2 + y^2 - 20x + 6y + 25 = 0$
(*c*) $x^2 - 6x - 4y + 5 = 0$ (*d*) $2x^2 + y^2 - 4x + 4y + 6 = 0$
(*e*) $3x^2 + 2y^2 + 12x - 4y + 15 = 0$ (*f*) $(x-1)(y+2) = 1$
(*g*) $xy - 3x - 2y + 5 = 0$ [*Hint*: Compare (*f*).] (*h*) $4x^2 + y^2 + 8x + 4y + 4 = 0$
(*i*) $2x^2 - 8x - y + 11 = 0$ (*j*) $25x^2 + 16y^2 - 100x - 32y - 284 = 0$

Ans. (*a*) empty graph; (*b*) ellipse, center at $(2, -3)$; (*c*) parabola, vertex at $(3, -1)$; (*d*) single point $(1, -2)$; (*e*) empty graph; (*f*) hyperbola, center at $(1, -2)$; (*g*) hyperbola, center at $(2, 3)$; (*h*) ellipse, center at $(-1, 2)$; (*i*) parabola, vertex at $(2, 3)$; (*j*) ellipse, center at $(2, 1)$

24. Find the focus, directrix, and length of the latus rectum of the following parabolas: (*a*) $10x^2 = 3y$; (*b*) $2y^2 = 3x$; (*c*) $4y = x^2 + 4x + 8$; (*d*) $8y = -x^2$.

Ans. (*a*) focus at $(0, \frac{3}{40})$, directrix $y = -\frac{3}{40}$, latum rectum $\frac{3}{10}$; (*b*) focus at $(\frac{3}{8}, 0)$, directrix $x = -\frac{3}{8}$, latus rectum $\frac{3}{2}$; (*c*) focus at $(-2, 2)$, directrix $y = 0$, latus rectum 4; (*d*) focus at $(0, -2)$, directrix $y = 2$, latus rectum 8

25. Find an equation for each parabola satisfying the following conditions:
(*a*) Focus at $(0, -3)$, directrix $y = 3$ (*b*) Focus at $(6, 0)$, directrix $x = 2$
(*c*) Focus at $(1, 4)$, directrix $y = 0$ (*d*) Vertex at $(1, 2)$ focus at $(1, 4)$
(*e*) Vertex at $(3, 0)$, directrix $x = 1$
(*f*) Vertex at the origin, y axis as axis of symmetry, contains the point $(3, 18)$
(*g*) Vertex at $(3, 5)$, axis of symmetry parallel to the y axis, contains the point $(5, 7)$
(*h*) Axis of symmetry parallel to the x axis, contains the points $(0, 1)$, $(3, 2)$, $(1, 3)$
(*i*) Latus rectum is the segment joining $(2, 4)$ and $(6, 4)$, contains the point $(8, 1)$
(*j*) Contains the points $(1, 10)$ and $(2, 4)$, axis of symmetry is vertical, vertex is on the line $4x - 3y = 6$

Ans. (*a*) $12y = -x^2$; (*b*) $8(x-4) = y^2$; (*c*) $8(y-2) = (x-1)^2$; (*d*) $8(y-2) = (x-1)^2$; (*e*) $8(x-3) = y^2$; (*f*) $y = 2x^2$; (*g*) $2(y-5) = (x-3)^2$; (*h*) $2(x - \frac{121}{40}) = -5(y - \frac{21}{10})^2$; (*i*) $4(y-5) = -(x-4)^2$; (*j*) $y - 2 = 2(x-3)^2$ or $y - \frac{2}{13} = 26(x - \frac{21}{13})^2$

26. Find an equation for each ellipse satisfying the following conditions:
(a) Center at the origin, one focus at $(0, 5)$, length of semimajor axis is 13
(b) Center at the origin, major axis on the y axis, contains the points $(1, 2\sqrt{3})$ and $(\frac{1}{2}, \sqrt{15})$
(c) Center at $(2, 4)$, focus at $(7, 4)$, contains the point $(5, 8)$
(d) Center at $(0, 1)$, one vertex at $(6, 1)$, eccentricity $\frac{2}{3}$
(e) Foci at $(0, \pm\frac{4}{3})$, contains $(\frac{4}{5}, 1)$
(f) Foci $(0, \pm 9)$, semiminor axis of length 12

Ans. (a) $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$; ($b$) $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$; ($c$) $\dfrac{(x-2)^2}{45} + \dfrac{(y-4)^2}{20} = 1$; ($d$) $\dfrac{x^2}{36} + \dfrac{(y-1)^2}{20} = 1$;
(e) $x^2 + \dfrac{9y^2}{25} = 1$; ($f$) $\dfrac{x^2}{144} + \dfrac{y^2}{225} = 1$

27. Find an equation for each hyperbola satisfying the following conditions:
(a) Center at the origin, transverse axis the x axis, contains the points $(6, 4)$ and $(-3, 1)$
(b) Center at the origin, one vertex at $(3, 0)$, one asymptote is $y = \frac{2}{3}x$
(c) Has asymptotes $y = \pm\sqrt{2}x$, contains the point $(1, 2)$
(d) Center at the origin, one focus at $(4, 0)$, one vertex at $(3, 0)$

Ans. (a) $\dfrac{5x^2}{36} - \dfrac{y^2}{4} = 1$; ($b$) $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$; ($c$) $\dfrac{y^2}{2} - x^2 = 1$; (d) $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$

28. Find an equation of the hyperbola consisting of all points $P(x, y)$ such that $|\overline{PF} - \overline{PF'}| = 2\sqrt{2}$, where $F = (\sqrt{2}, \sqrt{2})$ and $F' = (-\sqrt{2}, -\sqrt{2})$.

Ans. $xy = 1$

Chapter 6

Functions

FUNCTION OF A VARIABLE. A *function* is a rule that associates, with each value of a variable x in a certain set, exactly one value of another variable y. The variable y is then called the *dependent variable*, and x is called the *independent variable*. The set from which the values of x can be chosen is called the *domain* of the function. The set of all the corresponding values of y is called the *range* of the function.

EXAMPLE 1: The equation $x^2 - y = 10$, with x the independent variable, associates one value of y with each value of x. The function can be calculated with the formula $y = x^2 - 10$. The domain is the set of all real numbers. The same equation, $x^2 - y = 10$, with y taken as the independent variable, sometimes associates two values of x with each value of y. Thus, we must distinguish two functions of y: $x = \sqrt{10 + y}$ and $x = -\sqrt{10 + y}$. The domain of both these functions is the set of all y such that $y \geq -10$, since $\sqrt{10 + y}$ is not a real number when $10 + y < 0$.

If a function is denoted by a symbol f, then the expression $f(b)$ denotes the value obtained when f is applied to a number b in the domain of f. Often, a function is defined by giving the formula for an arbitrary value $f(x)$. For example, the formula $f(x) = x^2 - 10$ determines the first function mentioned in Example 1. The same function also can be defined by an equation like $y = x^2 - 10$.

EXAMPLE 2: (*a*) If $f(x) = x^3 - 4x + 2$, then

$$f(1) = (1)^3 - 4(1) + 2 = 1 - 4 + 2 = -1 \qquad f(-2) = (-2)^3 - 4(-2) + 2 = -8 + 8 + 2 = 2$$
$$f(a) = a^3 - 4a + 2$$

(*b*) The function $f(x) = 18x - 3x^2$ is defined for every number x; that is, without exception, $18x - 3x^2$ is a real number whenever x is a real number. Thus, the domain of the function is the set of all real numbers.
(*c*) The area A of a certain rectangle, one of whose sides has length x, is given by $A = 18x - 3x^2$. Here, both x and A must be positive. By completing the square, we obtain $A = -3(x - 3)^2 + 27$. In order to have $A > 0$, we must have $3(x - 3)^2 < 27$, which limits x to values below 6; hence, $0 < x < 6$. Thus, the function determining A has the open interval $(0, 6)$ as domain. From Fig. 6-1, we see that the range of the function is the interval $(0, 27]$.

Notice that the function of part (*c*) here is given by the same formula as the function of part (*b*), but the domain of the former is a proper subset of the domain of the latter.

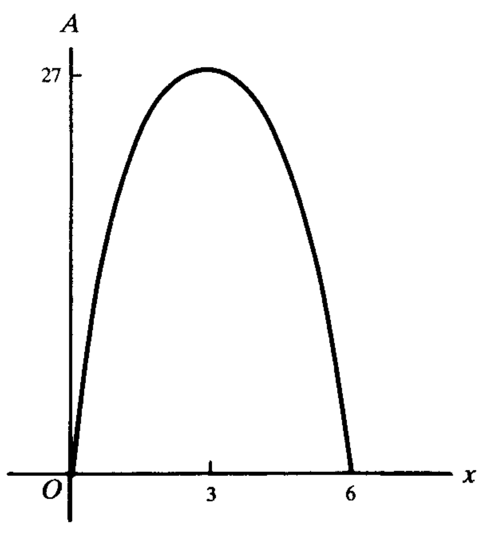


Fig. 6-1

THE GRAPH of a function f is the graph of the equation $y = f(x)$.

EXAMPLE 3: (*a*) Consider the function $f(x) = |x|$. Its graph is the graph of the equation $y = |x|$, shown in Fig. 6-2. Notice that $f(x) = x$ when $x \geq 0$, whereas $f(x) = -x$ when $x \leq 0$. The domain of f consists of all real numbers, but the range is the set of all nonnegative real numbers.
(b) The formula $g(x) = 2x + 3$ defines a function g. The graph of this function is the graph of the equation $y = 2x + 3$, which is the straight line with slope 2 and y intercept 3. The set of all real numbers is both the domain and range of g.

A function is said to be *defined on a set B* if it is defined for every point of that set.

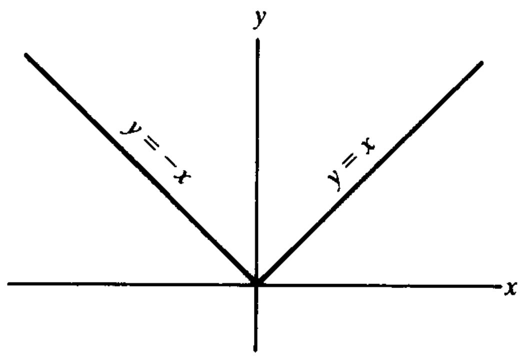


Fig. 6-2

Solved Problems

1. Given $f(x) = \dfrac{x-1}{x^2+2}$, find (*a*) $f(0)$; (*b*) $f(-1)$; (*c*) $f(2a)$; (*d*) $f(1/x)$; (*e*) $f(x+h)$.

(*a*) $f(0) = \dfrac{0-1}{0+2} = -\dfrac{1}{2}$ (*b*) $f(-1) = \dfrac{-1-1}{1+2} = -\dfrac{2}{3}$ (*c*) $f(2a) = \dfrac{2a-1}{4a^2+2}$

(*d*) $f(1/x) = \dfrac{1/x-1}{1/x^2+2} = \dfrac{x-x^2}{1+2x^2}$ (*e*) $f(x+h) = \dfrac{x+h-1}{(x+h)^2+2} = \dfrac{x+h-1}{x^2+2hx+h^2+2}$

2. If $f(x) = 2^x$, show that (*a*) $f(x+3) - f(x-1) = \frac{15}{2} f(x)$ and (*b*) $\dfrac{f(x+3)}{f(x-1)} = f(4)$.

(*a*) $f(x+3) - f(x-1) = 2^{x+3} - 2^{x-1} = 2^x(2^3 - \frac{1}{2}) = \frac{15}{2} f(x)$ (*b*) $\dfrac{f(x+3)}{f(x-1)} = \dfrac{2^{x+3}}{2^{x-1}} = 2^4 = f(4)$.

3. Determine the domains of the functions

(*a*) $y = \sqrt{4-x^2}$; (*b*) $y = \sqrt{x^2-16}$; (*c*) $y = \dfrac{1}{x-2}$;

(*d*) $y = \dfrac{1}{x^2-9}$; (*e*) $y = \dfrac{x}{x^2+4}$.

(*a*) Since y must be real, $4 - x^2 \geq 0$, or $x^2 \leq 4$. The domain is the interval $-2 \leq x \leq 2$.
(*b*) Here, $x^2 - 16 \geq 0$, or $x^2 \geq 16$. The domain consists of the intervals $x \leq -4$ and $x \geq 4$.
(*c*) The function is defined for every value of x except 2.
(*d*) The function is defined for $x \neq \pm 3$.
(*e*) Since $x^2 + 4 \neq 0$ for all x, the domain is the set of all real numbers.

4. Sketch the graph of the function defined as follows:

$$f(x) = 5 \text{ when } 0 < x \le 1 \qquad f(x) = 10 \text{ when } 1 < x \le 2$$
$$f(x) = 15 \text{ when } 2 < x \le 3 \qquad f(x) = 20 \text{ when } 3 < x \le 4 \qquad \text{etc.}$$

Determine the domain and range of the function.

The graph is shown in Fig. 6-3. The domain is the set of all positive real numbers, and the range is the set of integers, 5, 10, 15, 20, ...

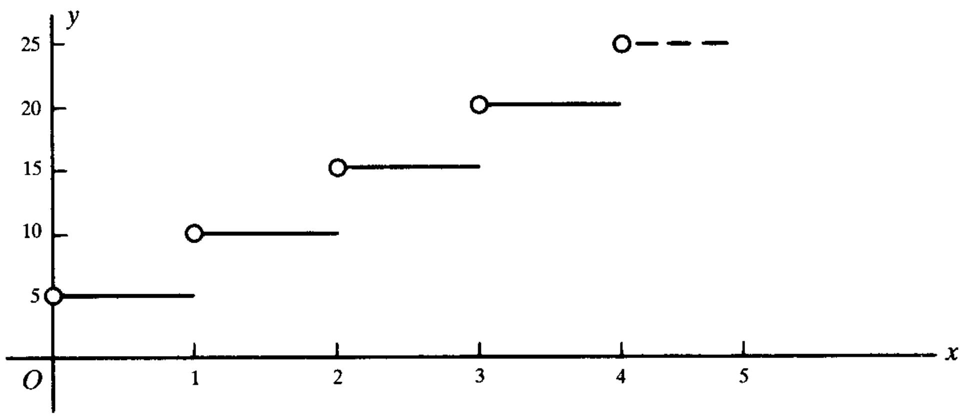


Fig. 6-3

5. A rectangular plot requires 2000 ft of fencing to enclose it. If one of its dimensions is x (in feet), express its area y (in square feet) as a function of x, and determine the domain of the function.

Since one dimension is x, the other is $\frac{1}{2}(2000 - 2x) = 1000 - x$. The area is then $y = x(1000 - x)$, and the domain of this function is $0 < x < 1000$.

6. Express the length l of a chord of a circle of radius 8 in as a function of its distance x (in inches) from the center of the circle. Determine the domain of the function.

From Fig. 6-4 we see that $\frac{1}{2}l = \sqrt{64 - x^2}$, so that $l = 2\sqrt{64 - x^2}$. The domain is the interval $0 \le x < 8$.

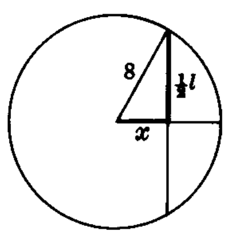


Fig. 6-4

7. From each corner of a square of tin, 12 in on a side, small squares of side x (in inches) are removed, and the edges are turned up to form an open box (Fig. 6-5). Express the volume V of the box (in cubic inches) as a function of x, and determine the domain of the function.

The box has a square base of side $12 - 2x$ and a height of x. The volume of the box is then $V = x(12 - 2x)^2 = 4x(6 - x)^2$. The domain is the interval $0 < x < 6$.

As x increases over its domain, V increases for a time and then decreases thereafter. Thus, among such boxes that may be constructed, there is one of greatest volume, say M. To determine M, it is necessary to locate the precise value of x at which V ceases to increase. This problem will be studied in a later chapter.

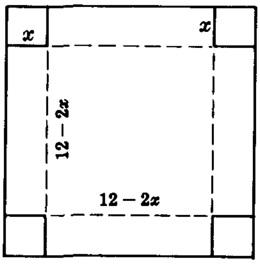


Fig. 6-5

8. If $f(x) = x^2 + 2x$, find $\dfrac{f(a+h) - f(a)}{h}$ and interpret the result.

$$\frac{f(a+h) - f(a)}{h} = \frac{[(a+h)^2 + 2(a+h)] - (a^2 + 2a)}{h} = 2a + 2 + h$$

On the graph of the function (Fig. 6-6), locate points P and Q whose respective abscissas are a and $a + h$. The ordinate of P is $f(a)$, and that of Q is $f(a+h)$. Then

$$\frac{f(a+h) - f(a)}{h} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \text{slope of } PQ$$

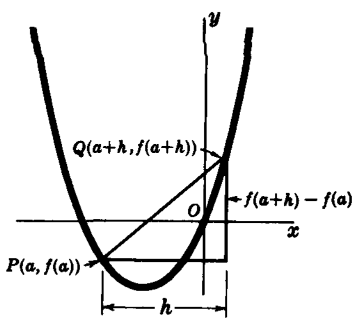


Fig. 6-6

9. Let $f(x) = x^2 - 2x + 3$. Evaluate (*a*) $f(3)$; (*b*) $f(-3)$; (*c*) $f(-x)$; (*d*) $f(x+2)$; (*e*) $f(x-2)$; (*f*) $f(x+h)$; (*g*) $f(x+h) - f(x)$; (*h*) $\dfrac{f(x+h) - f(x)}{h}$.

(*a*) $f(3) = 3^2 - 2(3) + 3 = 9 - 6 + 3 = 6$ (*b*) $f(-3) = (-3)^2 - 2(-3) + 3 = 9 + 6 + 3 = 18$
(*c*) $f(-x) = (-x)^2 - 2(-x) + 3 = x^2 + 2x + 3$
(*d*) $f(x+2) = (x+2)^2 - 2(x+2) + 3 = x^2 + 4x + 4 - 2x - 4 + 3 = x^2 + 2x + 3$
(*e*) $f(x-2) = (x-2)^2 - 2(x-2) + 3 = x^2 - 4x + 4 - 2x + 4 + 3 = x^2 - 6x + 11$
(*f*) $f(x+h) = (x+h)^2 - 2(x+h) + 3 = x^2 + 2hx + h^2 - 2x - 2h + 3 = x^2 + (2h-2)x + (h^2 - 2h + 3)$
(*g*) $f(x+h) - f(x) = [x^2 + (2h-2)x + (h^2 - 2h + 3)] - (x^2 - 2x + 3) = 2hx + h^2 - 2h = h(2x + h - 2)$
(*h*) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(2x + h - 2)}{h} = 2x + h - 2$

10. Draw the graph of the function $f(x) = \sqrt{4 - x^2}$, and find the domain and range of the function.

The graph of f is the graph of the equation $y = \sqrt{4 - x^2}$. For points on this graph, $y^2 = 4 - x^2$; that is, $x^2 + y^2 = 4$. The graph of the last equation is the circle with center at the origin and radius 2. Since

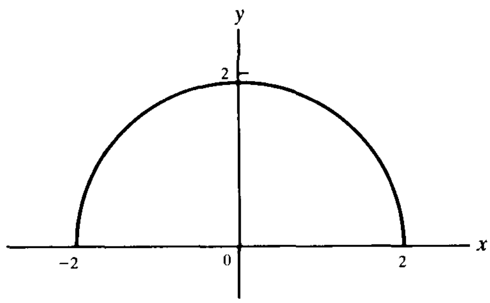


Fig. 6-7

$y = \sqrt{4 - x^2} \geq 0$, the desired graph is the upper half of that circle. Figure 6-7 shows that the domain is the interval $-2 \leq x \leq 2$, and the range is the interval $0 \leq y \leq 2$.

Supplementary Problems

11. If $f(x) = x^2 - 4x + 6$, find (*a*) $f(0)$; (*b*) $f(3)$; (*c*) $f(-2)$. Show that $f(\frac{1}{2}) = f(\frac{7}{2})$ and $f(2-h) = f(2+h)$. *Ans.* (*a*) -6; (*b*) 3; (*c*) 18

12. If $f(x) = \dfrac{x-1}{x+1}$, find (*a*) $f(0)$; (*b*) $f(1)$; (*c*) $f(-2)$. Show that $f\left(\dfrac{1}{x}\right) = -f(x)$ and $f\left(-\dfrac{1}{x}\right) = -\dfrac{1}{f(x)}$.
Ans. (*a*) -1; (*b*) 0; (*c*) 3

13. If $f(x) = x^2 - x$, show that $f(x+1) = f(-x)$.

14. If $f(x) = 1/x$, show that $f(a) - f(b) = f\left(\dfrac{ab}{b-a}\right)$.

15. If $y = f(x) = \dfrac{5x+3}{4x-5}$, show that $x = f(y)$.

16. Determine the domain of each of the following functions:

(*a*) $y = x^2 + 4$ (*b*) $y = \sqrt{x^2+4}$ (*c*) $y = \sqrt{x^2-4}$ (*d*) $y = \dfrac{x}{x+3}$

(*e*) $y = \dfrac{2x}{(x-2)(x+1)}$ (*f*) $y = \dfrac{1}{\sqrt{9-x^2}}$ (*g*) $y = \dfrac{x^2-1}{x^2+1}$ (*h*) $y = \sqrt{\dfrac{x}{2-x}}$

Ans. (*a*), (*b*), (*g*) all values of x; (*c*) $|x| \geq 2$; (*d*) $x \neq -3$; (*e*) $x \neq -1, 2$; (*f*) $-3 < x < 3$; (*h*) $0 \leq x < 2$

17. Compute $\dfrac{f(a+h) - f(a)}{h}$ in the following cases: (*a*) $f(x) = \dfrac{1}{x-2}$ when $a \neq 2$ and $a + h \neq 2$; (*b*) $f(x) = \sqrt{x-4}$ when $a \geq 4$ and $a + h \geq 4$; (*c*) $f(x) = \dfrac{x}{x+1}$ when $a \neq -1$ and $a + h \neq -1$.

Ans. (*a*) $\dfrac{-1}{(a-2)(a+h-2)}$; (*b*) $\dfrac{1}{\sqrt{a+h-4} + \sqrt{a-4}}$; (*c*) $\dfrac{1}{(a+1)(a+h+1)}$

18. Draw the graphs of the following functions, and find their domains and ranges:

(*a*) $f(x) = -x^2 + 1$ (*b*) $f(x) = \begin{cases} x - 1 & \text{if } 0 < x < 1 \\ 2x & \text{if } 1 \leq x \end{cases}$

(*c*) $f(x) = [x] =$ the greatest integer less than or equal to x

(*d*) $f(x) = \dfrac{x^2 - 4}{x - 2}$ (*e*) $f(x) = 5 - x^2$ (*f*) $f(x) = -4\sqrt{x}$

(*g*) $f(x) = |x - 3|$ (*h*) $f(x) = 4/x$ (*i*) $f(x) = |x|/x$

(*j*) $f(x) = x - |x|$ (*k*) $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$

Ans. (*a*) domain, all numbers; range, $y \leq 1$
(*b*) domain, $x > 0$; range, $-1 < y < 0$ or $y \geq 2$
(*c*) domain, all numbers; range, all integers
(*d*) domain, $x \neq 2$; range, $y \neq 4$
(*e*) domain, all numbers; range, $y \leq 5$
(*f*) domain, $x \geq 0$; range, $y \leq 0$
(*g*) domain, all numbers; range, $y \geq 0$
(*h*) domain, $x \neq 0$; range, $y \neq 0$
(*i*) domain, $x \neq 0$; range, $\{-1, 1\}$
(*j*) domain, all numbers; range, $y \leq 0$
(*k*) domain, all numbers; range, $y \geq 0$

19. Evaluate the expression $\dfrac{f(x+h) - f(x)}{h}$ for the following functions f: (*a*) $f(x) = 3x - x^2$; (*b*) $f(x) = \sqrt{2x}$; (*c*) $f(x) = 3x - 5$; (*d*) $f(x) = x^3 - 2$.

Ans. (*a*) $3 - 2x - h$; (*b*) $\dfrac{2}{\sqrt{2(x+h)} + \sqrt{2x}}$; (*c*) 3; (*d*) $3x^2 + 3xh + h^2$

20. Find a formula for the function f whose graph consists of all points (x, y) satisfying each of the following equations (in plain language, solve each equation for y): (*a*) $x^5y + 4x - 2 = 0$; (*b*) $x = \dfrac{2+y}{2-y}$; (*c*) $4x^2 - 4xy + y^2 = 0$.

Ans. (*a*) $f(x) = \dfrac{2 - 4x}{x^5}$; (*b*) $f(x) = \dfrac{2(x-1)}{x+1}$; (*c*) $f(x) = 2x$

21. (*a*) Prove the *vertical-line test*: A set of points in the xy plane is the graph of a function if and only if the set intersects every vertical line in at most one point. (*b*) Determine whether each set of points in Fig. 6-8 is the graph of a function.

Ans. only (*b*) is a function

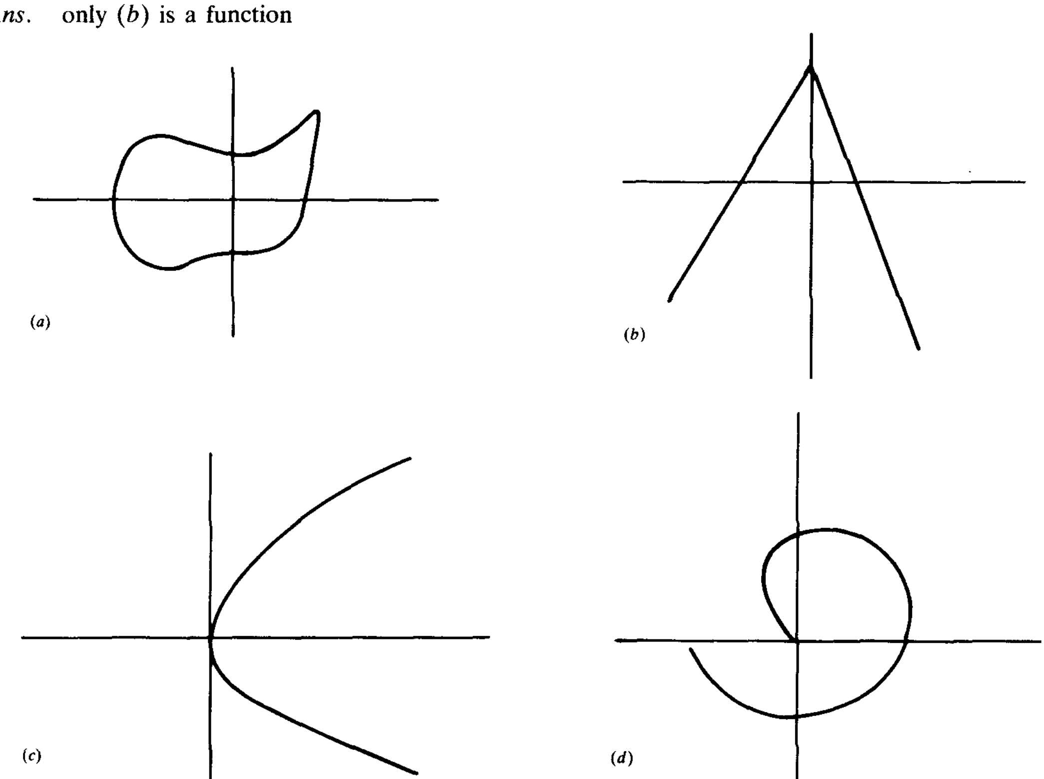


Fig. 6-8

Chapter 7

Limits

AN INFINITE SEQUENCE is a function whose domain is the set of positive integers. For example, when n is given in turn the values 1, 2, 3, 4, ..., the function defined by the formula $\frac{1}{n+1}$ yields the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$. The sequence is called an *infinite sequence* to indicate that there is no last term.

By the *general* or *nth term* of an infinite sequence we mean a formula s_n for the value of the function determining the sequence. The infinite sequence itself is often denoted by enclosing the general term in braces, as in $\{s_n\}$, or by displaying the first few terms of the sequence. For example, the general term s_n of the sequence in the preceding paragraph is $\frac{1}{n+1}$, and that sequence can be denoted by $\left\{\frac{1}{n+1}\right\}$ or by $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$.

LIMIT OF A SEQUENCE. If the terms of a sequence $\{s_n\}$ approach a fixed number c as n gets larger and larger, we say that c is the *limit* of the sequence, and we write either $s_n \to c$ or $\lim\limits_{n\to+\infty} s_n = c$.

As an example, consider the sequence

$$1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \ldots, 2 - \frac{1}{n}, \ldots \tag{7.1}$$

some of whose terms are plotted on the coordinate system in Fig. 7-1. As n increases, consecutive points cluster toward the point 2 in such a way that the distance of the points from 2 eventually becomes less than any positive number that might have been preassigned as a measure of closeness, however small. For example, the point $2 - \frac{1}{1001} = \frac{2001}{1001}$ and all subsequent points are at a distance less than $\frac{1}{1000}$ from 2, the point $\frac{20000001}{10000001}$ and all subsequent points are at a distance less than $\frac{1}{10000000}$ from 2, and so on. Hence, $\left\{2 - \frac{1}{n}\right\} \to 2$ or $\lim\limits_{n\to+\infty}\left(2 - \frac{1}{n}\right) = 2$.

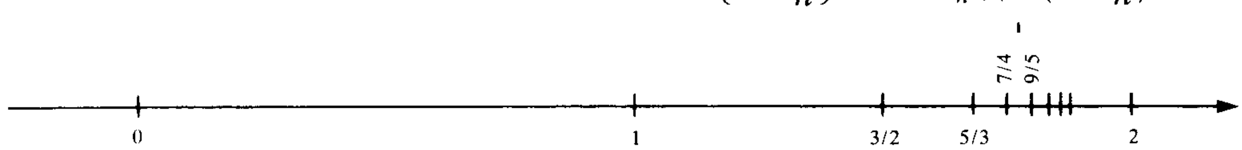


Fig. 7-1

The sequence (*7.1*) does not contain its limit 2 as a term. On the other hand, the sequence $1, \frac{1}{2}, 1, \frac{3}{4}, 1, \frac{5}{6}, 1, \ldots$ has 1 as limit, and every odd-numbered term is 1. Thus, a sequence having a limit may or may not contain that limit as a term.

Many sequences do not have a limit. For example, the sequence $\{(-1)^n\}$, that is, $-1, 1, -1, 1, -1, 1, \ldots$, keeps alternating between -1 and 1 and does not get closer and closer to any fixed number.

LIMIT OF A FUNCTION. If f is a function, then we say that $\lim\limits_{x\to a} f(x) = A$ if the value of $f(x)$ gets arbitrarily close to A as x gets closer and closer to a. For example, $\lim\limits_{x\to 3} x^2 = 9$, since x^2 gets arbitrarily close to 9 as x approaches as close as one wishes to 3.

The definition can be stated more precisely as follows: $\lim\limits_{x\to a} f(x) = A$ if and only if, for any chosen positive number ϵ, however small, there exists a positive number δ such that, whenever $0 < |x - a| < \delta$, then $|f(x) - A| < \epsilon$.

The gist of the definition is illustrated in Fig. 7-2: After ε has been chosen [that is, after interval (ii) has been chosen], then δ can be found [that is, interval (i) can be determined] so that, whenever $x \neq a$ is on interval (i), say at x_0, then $f(x)$ is on interval (ii), at $f(x_0)$. Notice the important fact that whether or not $\lim_{x \to a} f(x) = A$ is true does not depend upon the value of $f(x)$ when $x = a$. In fact, $f(x)$ need not even be defined when $x = a$.

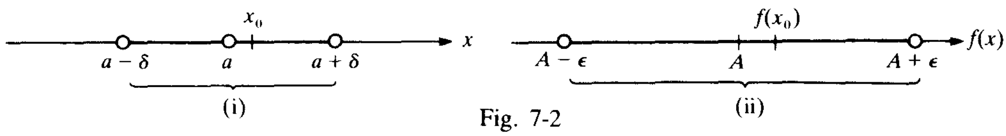


Fig. 7-2

EXAMPLE 1: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, although $\frac{x^2 - 4}{x - 2}$ is not defined when $x = 2$. Since $\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2$, we see that $\frac{x^2 - 4}{x - 2}$ approaches 4 as x approaches 2.

EXAMPLE 2: Let us use the precise definition to show that $\lim_{x \to 2} (x^2 + 3x) = 10$. Let $\epsilon > 0$ be chosen. We must produce a $\delta > 0$ such that, whenever $0 < |x - 2| < \delta$ then $|(x^2 + 3x) - 10| < \epsilon$. First we note that

$$|(x^2 + 3x) - 10| = |(x - 2)^2 + 7(x - 2)| \leq |x - 2|^2 + 7|x - 2|$$

Also, if $0 < \delta \leq 1$, then $\delta^2 \leq \delta$. Hence, if we take δ to be the minimum of 1 and $\epsilon/8$, then, whenever $0 < |x - 2| < \delta$,

$$|(x^2 + 3x) - 10| < \delta^2 + 7\delta \leq \delta + 7\delta = 8\delta \leq \epsilon$$

The definition of $\lim_{x \to a} f(x) = A$ given above is equivalent to the following definition in terms of infinite sequences: $\lim_{x \to a} f(x) = A$ if and only if, for any sequence $\{s_n\}$ such that $\lim_{n \to +\infty} s_n = a$, $\lim_{n \to +\infty} f(s_n) = A$. In other words, no matter what sequence $\{s_n\}$ we may consider such that s_n approaches a, the corresponding sequence $\{f(s_n)\}$ must approach A.

RIGHT AND LEFT LIMITS. By $\lim_{x \to a^-} f(x) = A$ we mean that $f(x)$ approaches A as x approaches a through values less than a, that is, as x approaches *a from the left*. Similarly, $\lim_{x \to a^+} f(x) = A$ means that $f(x)$ approaches A as x approaches a through values greater than a, that is, as x approaches *a from the right*. The statement $\lim_{x \to a} f(x) = A$ is equivalent to the conjunction of the two statements $\lim_{x \to a^-} f(x) = A$ and $\lim_{x \to a^+} f(x) = A$. The existence of the limit from the left does not imply the existence of the limit from the right, and conversely.

When a function f is defined on only one side of a point a, then $\lim_{x \to a} f(x)$ is identical with the one-sided limit, if it exists. For example, if $f(x) = \sqrt{x}$, then f is defined only to the right of zero. Hence, $\lim_{x \to 0} \sqrt{x} = \lim_{x \to 0^+} \sqrt{x} = 0$. Of course, $\lim_{x \to 0^-} \sqrt{x}$ does not exist, since $\sqrt{x}$ is not defined when $x < 0$. On the other hand, consider the function $g(x) = \sqrt{1/x}$, which is defined only for $x > 0$. In this case, $\lim_{x \to 0^+} \sqrt{1/x}$ does not exist and, therefore, $\lim_{x \to 0} \sqrt{1/x}$ does not exist.

EXAMPLE 3: The function $f(x) = \sqrt{9 - x^2}$ has the interval $-3 \leq x \leq 3$ as its domain of definition. If a is any number on the open interval $-3 < x < 3$, then $\lim_{x \to a} \sqrt{9 - x^2}$ exists and is equal to $\sqrt{9 - a^2}$. Now consider $a = 3$. First, let x approach 3 from the left; then $\lim_{x \to 3^-} \sqrt{9 - x^2} = 0$. Next, let x approach 3 from the right; then $\lim_{x \to 3^+} \sqrt{9 - x^2}$ does not exist, since for $x > 3$, $\sqrt{9 - x^2}$ is not a real number. Thus, $\lim_{x \to 3} \sqrt{9 - x^2} = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0$.

Similarly, $\lim_{x \to -3^+} \sqrt{9 - x^2}$ exists and is equal to 0, but $\lim_{x \to -3^-} \sqrt{9 - x^2}$ does not exist. Thus, $\lim_{x \to -3} \sqrt{9 - x^2} = 0$.

THEOREMS ON LIMITS. The following theorems on limits are listed for future reference.

Theorem 7.1: If $f(x) = c$, a constant, then $\lim_{x \to a} f(x) = c$.

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, then:

Theorem 7.2: $\lim_{x \to a} kf(x) = kA$, k being any constant.

Theorem 7.3: $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$.

Theorem 7.4: $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = AB$.

Theorem 7.5: $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{A}{B}$, provided $B \neq 0$.

Theorem 7.6: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}$, provided $\sqrt[n]{A}$ is a real number.

INFINITY. We say that a sequence $\{s_n\}$ approaches $+\infty$, and we write $s_n \to +\infty$ or $\lim_{n \to +\infty} s_n = +\infty$, if the values s_n eventually become and thereafter remain greater than any preassigned positive number, however large. For example, $\lim_{n \to +\infty} \sqrt{n} = +\infty$ and $\lim_{n \to +\infty} n^2 = +\infty$.

We say that a sequence $\{s_n\}$ approaches $-\infty$, and we write $s_n \to -\infty$ or $\lim_{n \to +\infty} s_n = -\infty$, if the values s_n eventually become and thereafter remain less than any preassigned negative number, however small. For example, $\lim_{n \to +\infty} -n = -\infty$ and $\lim_{n \to +\infty} (10 - n^2) = -\infty$.

The corresponding notions for functions are the following:

We say that $f(x)$ approaches $+\infty$ as x approaches a, and we write $\lim_{x \to a} f(x) = +\infty$, if, as x approaches its limit a (without assuming the value a), $f(x)$ eventually becomes and thereafter remains greater than any preassigned positive number, however large. This can be given the following more precise definition: $\lim_{x \to a} f(x) = +\infty$ if and only if, for any positive number M, there exists a positive number δ such that, whenever $0 < |x - a| < \delta$, then $f(x) > M$.

We say that $f(x)$ approaches $-\infty$ as x approaches a, and we write $\lim_{x \to a} f(x) = -\infty$, if, as x approaches its limit a (without assuming the value a), $f(x)$ eventually becomes and thereafter remains less than any preassigned negative number. By $\lim_{x \to a} f(x) = \infty$ we mean that, as x approaches its limit a (without assuming the value a), $|f(x)|$ eventually becomes and thereafter remains larger than any preassigned number. Thus, $\lim_{x \to a} f(x) = \infty$ if and only if $\lim_{x \to a} |f(x)| = +\infty$.

EXAMPLE 4: (*a*) $\lim_{x \to 0} \dfrac{1}{x^2} = +\infty$ (*b*) $\lim_{x \to 1} \dfrac{-1}{(x-1)^2} = -\infty$ (*c*) $\lim_{x \to 0} \dfrac{1}{x} = \infty$

These ideas can be extended to one-sided (left and right) limits in the obvious way.

EXAMPLE 5: (*a*) $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$, since, as x approaches 0 from the right (that is, through positive numbers) $\dfrac{1}{x}$ is positive and eventually becomes larger than any preassinged number.

(*b*) $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$, since, as x approaches 0 from the left (that is, through negative numbers), $\dfrac{1}{x}$ is negative and eventually becomes smaller than any preassigned number.

The limit concepts already introduced also can be extended in an obvious way to the case in which the variable approaches $+\infty$ or $-\infty$. For example, $\lim_{x \to +\infty} f(x) = A$ means that $f(x)$ approaches A as $x \to +\infty$; or, in more precise terms, given any positive ϵ, there exists a number N such that, whenever $x > N$, $|f(x) - A| < \epsilon$.

Similar definitions can be given for the statements $\lim_{x \to -\infty} f(x) = A$, $\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to +\infty} f(x) = -\infty$, and $\lim_{x \to -\infty} f(x) = +\infty$.

EXAMPLE 6: $\lim_{x\to+\infty} \frac{1}{x} = 0$ and $\lim_{x\to+\infty}\left(2+\frac{1}{x^2}\right) = 2$.

Caution: When $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, Theorems 3.3 to 3.5 do not make sense and cannot be used. For example, $\lim_{x\to 0}\frac{1}{x^2} = +\infty$ and $\lim_{x\to 0}\frac{1}{x^4} = +\infty$; however, $\lim_{x\to 0}\frac{1/x^2}{1/x^4} = \lim_{x\to 0} x^2 = 0$.

Solved Problems

1. Write the first five terms of each of the following sequences.

(*a*) $\left\{1-\frac{1}{2n}\right\}$: Set $s_n = 1-\frac{1}{2n}$; then $s_1 = 1-\frac{1}{2\cdot 1} = \frac{1}{2}$, $s_2 = 1-\frac{1}{2\cdot 2} = \frac{3}{4}$, $s_3 = 1-\frac{1}{2\cdot 3} = \frac{5}{6}$, $s_4 = 1-\frac{1}{2\cdot 4} = \frac{7}{8}$, and $s_5 = \frac{9}{10}$. The required terms are $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$.

(*b*) $\left\{(-1)^{n+1}\frac{1}{3n-1}\right\}$: Here $s_1 = (-1)^2\frac{1}{3\cdot 1-1} = \frac{1}{2}$, $s_2 = (-1)^3\frac{1}{3\cdot 2-1} = -\frac{1}{5}$. $s_3 = (-1)^4\frac{1}{3\cdot 3-1} = \frac{1}{8}$, $s_4 = -\frac{1}{11}$, $s_5 = \frac{1}{14}$. The required terms are $\frac{1}{2}, -\frac{1}{5}, \frac{1}{8}, -\frac{1}{11}, \frac{1}{14}$.

(*c*) $\left\{\frac{2n}{1+n^2}\right\}$: The terms are $1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \frac{5}{13}$.

(*d*) $\left\{(-1)^{n+1}\frac{n}{(n+1)(n+2)}\right\}$: The terms are $\frac{1}{2\cdot 3}, \frac{-2}{3\cdot 4}, \frac{3}{4\cdot 5}, \frac{-4}{5\cdot 6}, \frac{5}{6\cdot 7}$.

(*e*) $\{\frac{1}{2}[(-1)^n+1]\}$: The terms are 0, 1, 0, 1, 0.

2. Write the general term of each of the following sequences.

(*a*) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$: The terms are the reciprocals of the odd positive integers. The general term is $\frac{1}{2n-1}$.

(*b*) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$: Apart from sign, these are the reciprocals of the positive integers. The general term is $(-1)^{n+1}\frac{1}{n}$ or $(-1)^{n-1}\frac{1}{n}$.

(*c*) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$: The terms are the reciprocals of the squares of the positive integers. The general term is $1/n^2$.

(*d*) $\frac{1}{2}, \frac{1\cdot 3}{2\cdot 4}, \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}, \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}, \ldots$: The general term is $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$.

(*e*) $\frac{1}{2}, -\frac{4}{9}, \frac{9}{28}, -\frac{16}{65}, \ldots$: Apart from sign, the numerators are the squares of positive integers and the denominators are the cubes of these integers increased by 1. The general term is $(-1)^{n+1}\frac{n^2}{n^3+1}$.

3. Determine the limit of each of the following sequences.

(*a*) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$: The general term is $1/n$. As n takes on the values 1, 2, 3, 4, ... in turn, $1/n$ decreases but remains positive. The limit is 0.
(*b*) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$: The general term is $(1/n)^2$; the limit is 0.
(*c*) $2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, \ldots$: The general term is $3-1/n$; the limit is 3.
(*d*) $5, 4, \frac{11}{3}, \frac{7}{2}, \frac{17}{5}, \ldots$: The general term is $3+2/n$; the limit is 3.
(*e*) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$: The general term is $1/2^n$; the limit is 0.
(*f*) 0.9, 0.99, 0.999, 0.9999, 0.99999, ...: The general term is $1-1/10^n$; the limit is 1.

4. Evaluate the limit in each of the following.

(*a*) $\lim_{x\to 2} 5x = 5 \lim_{x\to 2} x = 5\cdot 2 = 10$

(*b*) $\lim_{x\to 2} (2x+3) = 2\lim_{x\to 2} x + \lim_{x\to 2} 3 = 2\cdot 2 + 3 = 7$

(*c*) $\lim_{x\to 2} (x^2 - 4x + 1) = 4 - 8 + 1 = -3$

(*d*) $\lim_{x\to 3} \dfrac{x-2}{x+2} = \dfrac{\lim_{x\to 3}(x-2)}{\lim_{x\to 3}(x+2)} = \dfrac{1}{5}$

(*e*) $\lim_{x\to -2} \dfrac{x^2-4}{x^2+4} = \dfrac{4-4}{4+4} = 0$

(*f*) $\lim_{x\to 4} \sqrt{25-x^2} = \sqrt{\lim_{x\to 4}(25-x^2)} = \sqrt{9} = 3$

Note: Do not assume from these problems that $\lim_{x\to a} f(x)$ is invariably $f(a)$.

(*g*) $\lim_{x\to -5} \dfrac{x^2-25}{x+5} = \lim_{x\to -5}(x-5) = -10$

5. Examine the behavior of $f(x) = (-1)^x$ as x ranges over the sequences (*a*) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$ and (*b*) $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots$. (*c*) What can be said concerning $\lim_{x\to 0} (-1)^x$ and $f(0)$?

(*a*) $(-1)^x \to -1$ over the sequence $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$.
(*b*) $(-1)^x \to +1$ over the sequence $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots$.
(*c*) Since $(-1)^x$ approaches different limits over the two sequences, $\lim_{x\to 0}(-1)^x$ does not exist; $f(0) = (-1)^0 = +1$.

6. Evaluate the limit in each of the following.

(*a*) $\lim_{x\to 4} \dfrac{x-4}{x^2-x-12} = \lim_{x\to 4} \dfrac{x-4}{(x+3)(x-4)} = \lim_{x\to 4} \dfrac{1}{x+3} = \dfrac{1}{7}$

The division by $x-4$ before passing to the limit is valid since $x \neq 4$ as $x \to 4$; hence, $x-4$ is never zero.

(*b*) $\lim_{x\to 3} \dfrac{x^3-27}{x^2-9} = \lim_{x\to 3} \dfrac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x\to 3} \dfrac{x^2+3x+9}{x+3} = \dfrac{9}{2}$

(*c*) $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \dfrac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0} \dfrac{2hx+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$

Here, and again in Problems 8 and 9, h is a variable so that it could be argued that we are in reality dealing with functions of two variables. However, the fact that x is a variable plays no role in these problems; we may then for the moment consider x to be a constant, that is, some one of the values of its range. The gist of the problem, as we shall see in Chapter 9, is that if x is any value, say $x = x_0$, in the domain of $y = x^2$, then $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$ is always twice the selected value of x.

(*d*) $\lim_{x\to 2} \dfrac{4-x^2}{3-\sqrt{x^2+5}} = \lim_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})} = \lim_{x\to 2} \dfrac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2}$

$= \lim_{x\to 2}(3+\sqrt{x^2+5}) = 6$

(*e*) $\lim_{x\to 1} \dfrac{x^2+x-2}{(x-1)^2} = \lim_{x\to 1} \dfrac{(x-1)(x+2)}{(x-1)^2} = \lim_{x\to 1} \dfrac{x+2}{x-1} = \infty$; no limit exists.

7. In the following, interpret $\lim_{x\to \pm\infty}$ as an abbreviation for $\lim_{x\to +\infty}$ or $\lim_{x\to -\infty}$. Evaluate the limit by first dividing numerator and denominator by the highest power of x present and then using $\lim_{x\to\infty} \dfrac{1}{x} = 0$.

(*a*) $\lim_{x\to\infty} \dfrac{3x-2}{9x+7} = \lim_{x\to\infty} \dfrac{3-2/x}{9+7/x} = \dfrac{3-0}{9+0} = \dfrac{1}{3}$

(*b*) $\lim_{x\to\infty} \frac{6x^2+2x+1}{6x^2-3x+4} = \lim_{x\to\infty} \frac{6+2/x+1/x^2}{6-3/x+4/x^2} = \frac{6+0+0}{6-0+0} = 1$

(*c*) $\lim_{x\to\infty} \frac{x^2+x-2}{4x^3-1} = \lim_{x\to\infty} \frac{1/x+1/x^2-2/x^3}{4-1/x^3} = \frac{0}{4} = 0$

(*d*) $\lim_{x\to\infty} \frac{2x^3}{x^2+1}$: $\lim_{x\to-\infty} \frac{2}{1/x+1/x^3} = -\infty$; no limit exists

$\lim_{x\to+\infty} \frac{2}{1/x+1/x^3} = +\infty$; no limit exists

8. Given $f(x) = x^2 - 3x$, find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Since $f(x) = x^2 - 3x$, we have $f(x+h) = (x+h)^2 - 3(x+h)$ and

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x^2+2hx+h^2-3x-3h)-(x^2-3x)}{h} = \lim_{h\to 0} \frac{2hx+h^2-3h}{h}$$
$$= \lim_{h\to 0} (2x+h-3) = 2x-3$$

9. Given $f(x) = \sqrt{5x+1}$, find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ when $x > -\frac{1}{5}$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{5x+5h+1}-\sqrt{5x+1}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{5x+5h+1}-\sqrt{5x+1}}{h} \, \frac{\sqrt{5x+5h+1}+\sqrt{5x+1}}{\sqrt{5x+5h+1}+\sqrt{5x+1}}$$
$$= \lim_{h\to 0} \frac{(5x+5h+1)-(5x+1)}{h(\sqrt{5x+5h+1}+\sqrt{5x+1})}$$
$$= \lim_{h\to 0} \frac{5}{\sqrt{5x+5h+1}+\sqrt{5x+1}} = \frac{5}{2\sqrt{5x+1}}$$

10. In each of the following, determine the points $x = a$ for which each denominator is zero. Then examine y as $x \to a^-$ and $x \to a^+$.

(*a*) $y = f(x) = 2/x$: The denominator is zero when $x = 0$. As $x \to 0^-$, $y \to -\infty$; as $x \to 0^+$, $y \to +\infty$.

(*b*) $y = f(x) = \frac{x-1}{(x+3)(x-2)}$: the denominator is zero for $x = -3$ and $x = 2$. As $x \to -3^-$, $y \to -\infty$; as $x \to -3^+$, $y \to +\infty$. As $x \to 2^-$, $y \to -\infty$; as $x \to 2^+$, $y \to +\infty$.

(*c*) $y = f(x) = \frac{x-3}{(x+2)(x-1)}$: The denominator is zero for $x = -2$ and $x = 1$. As $x \to -2^-$, $y \to -\infty$; as $x \to -2^+$, $y \to +\infty$. As $x \to 1^-$, $y \to +\infty$; as $x \to 1^+$, $y \to -\infty$.

(*d*) $y = f(x) = \frac{(x+2)(x-1)}{(x-3)^2}$: The denominator is zero for $x = 3$. As $x \to 3^-$, $y \to +\infty$; as $x \to 3^+$, $y \to +\infty$.

(*e*) $y = f(x) = \frac{(x+2)(1-x)}{x-3}$: The denominator is zero for $x = 3$. As $x \to 3^-$, $y \to +\infty$; as $x \to 3^+$, $y \to -\infty$.

11. Examine (*a*) $\lim_{x\to 0} \frac{1}{3+2^{1/x}}$ and (*b*) $\lim_{x\to 0} \frac{1+2^{1/x}}{3+2^{1/x}}$.

(*a*) Let $x \to 0^-$; then $1/x \to -\infty$, $2^{1/x} \to 0$, and $\lim_{x\to 0^-} \frac{1}{3+2^{1/x}} = \frac{1}{3}$.

Let $x \to 0^+$; then $1/x \to +\infty$, $2^{1/x} \to +\infty$, and $\lim_{x\to 0^+} \frac{1}{3+2^{1/x}} = 0$.

Thus $\lim_{x\to 0} \frac{1}{3+2^{1/x}}$ does not exist.

(*b*) Let $x \to 0^-$; then $2^{1/x} \to 0$ and $\lim_{x\to 0^-} \frac{1+2^{1/x}}{3+2^{1/x}} = \frac{1}{3}$.

Let $x \to 0^+$. For $x \neq 0$, $\dfrac{1+2^{1/x}}{3+2^{1/x}} = \dfrac{2^{-1/x}+1}{3 \cdot 2^{-1/x}+1}$ and since $\lim_{x\to 0^+} 2^{-1/x} = 0$, $\lim_{x\to 0^+} \dfrac{2^{-1/x}+1}{3\cdot 2^{-1/x}+1} = 1$. Thus, $\lim_{x\to 0} \dfrac{1+2^{1/x}}{3+2^{1/x}}$ does not exist.

12. For each of the functions of Problem 10, examine y as $x \to -\infty$ and as $x \to +\infty$.

(*a*) When $|x|$ is large, $|y|$ is small.
For $x = -1000$, $y < 0$; as $x \to -\infty$, $y \to 0^-$. For $x = +1000$, $y > 0$; as $x \to +\infty$, $y \to 0^+$.
(*b*), (*c*) Same as (*a*).
(*d*) When $|x|$ is large, $|y|$ is approximately 1.
For $x = -1000$, $y < 1$; as $x \to -\infty$, $y \to 1^-$. For $x = +1000$, $y > 1$; as $x \to +\infty$, $y \to 1^+$.
(*e*) When $|x|$ is large, $|y|$ is large.
For $x = -1000$, $y > 0$; as $x \to -\infty$, $y \to +\infty$. For $x = +1000$, $y < 0$; as $x \to +\infty$, $y \to -\infty$.

13. Examine the function of Problem 4 in Chapter 6 as $x \to a^-$ and as $x \to a^+$ when a is any positive integer.

Consider, as a typical case, $a = 2$. As $x \to 2^-$, $f(x) \to 10$. As $x \to 2^+$, $f(x) \to 15$. Thus, $\lim_{x\to 2} f(x)$ does not exist. In general, the limit fails to exist for all positive integers. (Note, however, that $\lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = 5$, since $f(x)$ is not defined for $x \leq 0$.)

14. Use the precise definition to show that (*a*) $\lim_{x\to 1} (4x^3 + 3x^2 - 24x + 22) = 5$ and (*b*) $\lim_{x\to -1} (-2x^3 + 9x + 4) = -3$.

(*a*) Let ϵ be chosen. For $0 < |x-1| < \lambda < 1$,

$$\begin{aligned}|(4x^3 + 3x^2 - 24x + 22) - 5| &= |4(x-1)^3 + 15x^2 - 36x + 21| = |4(x-1)^3 + 15(x-1)^2 - 6(x-1)| \\ &\leq 4|x-1|^3 + 15|x-1|^2 + 6|x-1| \\ &< 4\lambda + 15\lambda + 6\lambda = 25\lambda\end{aligned}$$

Now $|(4x^3 + 3x^2 - 24x + 22) - 5| < \epsilon$ for $\lambda < \epsilon/25$; hence, any positive number smaller than both 1 and $\epsilon/25$ is an effective δ, and the limit is established.

(*b*) Let ϵ be chosen. For $0 < |x+1| < \lambda < 1$,

$$\begin{aligned}|(-2x^3 + 9x + 4) + 3| &= |-2(x+1)^3 + 6(x+1)^2 + 3(x+1)| \\ &\leq 2|x+1|^3 + 6|x+1|^2 + 3|x+1| < 11\lambda\end{aligned}$$

Any positive number smaller than both 1 and $\epsilon/11$ is an effective δ, and the limit is established.

15. Given $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, prove:
(*a*) $\lim_{x\to a} [f(x) + g(x)] = A + B$ (*b*) $\lim_{x\to a} f(x)g(x) = AB$ (*c*) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$, $B \neq 0$

Since $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, it follows by the precise definition that for numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$, however small, there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that:

$$\text{Whenever } 0 < |x-a| < \delta_1, \text{ then } |f(x) - A| < \epsilon_1 \tag{1}$$

$$\text{Whenever } 0 < |x-a| < \delta_2, \text{ then } |g(x) - B| < \epsilon_2 \tag{2}$$

Let λ denote the smaller of δ_1 and δ_2; now

$$\text{Whenever } 0 < |x-a| < \lambda, \text{ then } |f(x) - A| < \epsilon_1 \textit{ and } |g(x) - B| < \epsilon_2 \tag{3}$$

(*a*) Let ϵ be chosen. We are required to produce a $\delta > 0$ such that

$$\text{Whenever } 0 < |x-a| < \delta, \text{ then } |[f(x) + g(x)] - (A+B)| < \epsilon$$

Now $|[f(x) + g(x)] - (A+B)| = |[f(x) - A] + [g(x) - B]| \leq |f(x) - A| + |g(x) - B|$. By (*3*), $|f(x) - A| < \epsilon_1$ whenever $0 < |x-a| < \lambda$ and $|g(x) - A| < \epsilon_2$ whenever $0 < |x-a| < \lambda$, where λ is the smaller of δ_1 and δ_2. Thus,

$$|[f(x) + g(x)] - (A + B)| < \epsilon_1 + \epsilon_2 \text{ whenever } 0 < |x - a| < \lambda$$

Take $\epsilon_1 = \epsilon_2 = \frac{1}{2}\epsilon$ and $\delta = \lambda$ for this choice of ϵ_1 and ϵ_2; then, as required,

$$|[f(x) + g(x)] - (A + B)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \text{ whenever } 0 < |x - a| < \delta$$

(*b*) Let ϵ be chosen. We are required to produce a $\delta > 0$ such that

$$\text{Whenever } 0 < |x - a| < \delta \text{ then } |f(x)g(x) - AB| < \epsilon$$

Now
$$\begin{aligned} |f(x)g(x) - AB| &= |[f(x) - A][g(x) - B] + B[f(x) - A] + A[g(x) - B]| \\ &\le |f(x) - A||g(x) - B| + |B||f(x) - A| + |A||g(x) - B| \end{aligned}$$

so that, by (*3*), $|f(x)g(x) - AB| < \epsilon_1\epsilon_2 + |B|\epsilon_1 + |A|\epsilon_2$ whenever $0 < |x - a| < \lambda$. Take ϵ_1 and ϵ_2 such that $\epsilon_1\epsilon_2 < \frac{1}{3}\epsilon$, $\epsilon_1 < \frac{1}{3}\frac{\epsilon}{|B|}$, and $\epsilon_2 < \frac{1}{3}\frac{\epsilon}{|A|}$ are simultaneously satisfied and let $\delta = \lambda$ for this choice of ϵ_1 and ϵ_2. Then, as required,

$$|f(x)g(x) - AB| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ whenever } 0 < |x - a| < \delta$$

(*c*) Since $\frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)}$, the theorem follows from (*b*) provided we can show that $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{B}$, for $B \neq 0$.

Let ϵ be chosen. We are required to produce a $\delta > 0$ such that

$$\text{Whenever } 0 < |x - a| < \delta \text{ then } \left|\frac{1}{g(x)} - \frac{1}{B}\right| < \epsilon$$

Now
$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \left|\frac{B - g(x)}{Bg(x)}\right| = \frac{|g(x) - B|}{|B||g(x)|} = \frac{|g(x) - B|}{|B|}\frac{1}{|g(x)|}$$

By (*2*),
$$|g(x) - B| < \epsilon_2 \text{ whenever } 0 < |x - a| < \delta_2$$

However, we are also dealing with $1/g(x)$, so we must be sure δ_2 is sufficiently small that the interval $a - \delta_2 < x < a + \delta_2$ does not contain a root of $g(x) = 0$. Let $\delta_3 \le \delta_2$ meet this requirement so that $|g(x) - B| < \epsilon_2$ and $|g(x)| > 0$ whenever $0 < |x - a| \le \delta_3$. Now $|g(x)| > 0$ on the interval implies $|g(x)| > b > 0$ and $\frac{1}{|g(x)|} < \frac{1}{b}$ on the interval for some b. Thus, we have

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| < \frac{\epsilon_2}{|B|}\frac{1}{b} \text{ whenever } 0 < |x - a| < \delta_3$$

Take $\epsilon_2 < \epsilon b|B|$, so that $\frac{\epsilon_2}{|B|b} < \epsilon$ and $\delta = \delta_3$ for this choice of ϵ_2. Then, as required,

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

16. Prove (*a*) $\lim_{x\to 2^-} \frac{1}{(x-2)^3} = -\infty$; (*b*) $\lim_{x\to+\infty} \frac{x}{x+1} = 1$; (*c*) $\lim_{x\to+\infty} \frac{x^2}{x-1} = +\infty$.

(*a*) Let M be any negative number. Choose δ positive and equal to the minimum of 1 and $\frac{1}{|M|}$. Assume $x < 2$ and $0 < |x - 2| < \delta$. Then $|x - 2|^3 < \delta^3 \le \delta \le \frac{1}{|M|}$. Hence, $\frac{1}{|x-2|^3} > |M| = -M$. But $(x - 2)^3 < 0$. Therefore, $\frac{1}{(x-2)^3} = -\frac{1}{|x-2|^3} < M$.

(*b*) Let ϵ be any positive number, and let $M = 1/\epsilon$. Assume $x > M$. Then

$$\left|\frac{x}{x+1} - 1\right| = \left|\frac{1}{x+1}\right| = \frac{1}{x+1} < \frac{1}{x} < \frac{1}{M} = \epsilon$$

(*c*) Let $M > 1$ be any positive number. Assume $x > M$. Then $\frac{x^2}{x-1} \ge \frac{x^2}{x} = x > M$.

Supplementary Problems

17. Write the first five terms of each sequence:

(a) $\left\{1+\frac{1}{n}\right\}$ (b) $\left\{\frac{1}{n(n+1)}\right\}$ (c) $\{a+(n-1)d\}$ (d) $\{(-1)^{n+1}ar^{n-1}\}$

(e) $\left\{\frac{n}{\sqrt{1+n^2}}\right\}$ (f) $\left\{\frac{\sqrt{n+1}}{n}\right\}$ (g) $\left\{(-1)^{n+1}\frac{n!}{n^n}\right\}$ (h) $\left\{\frac{(2n)!}{3^n 5^{n-1}}\right\}$

Ans. (a) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$; (b) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$; (c) $a, a+d, a+2d, a+3d, a+4d$; (d) $a, -ar, ar^2, -ar^3, ar^4$; (e) $1/\sqrt{2}, 2/\sqrt{5}, 3/\sqrt{10}, 4/\sqrt{17}, 5/\sqrt{26}$; (f) $\sqrt{2}, \frac{1}{2}\sqrt{3}, \frac{2}{3}, \frac{1}{4}\sqrt{5}, \sqrt{6}/5$; (g) $1, -\frac{1}{2}, \frac{2}{9}, -\frac{3}{32}, \frac{24}{625}$; (h) $\frac{2}{3}, \frac{2^3}{3\cdot 5}, \frac{2^4}{3\cdot 5}, \frac{7\cdot 2^7}{3^2\cdot 5^2}, \frac{7\cdot 2^8}{3\cdot 5^2}$

18. Determine the general term of each sequence:

(a) $1/2, 2/3, 3/4, 4/5, 5/6, \ldots$ (b) $1/2, -1/6, 1/12, -1/20, 1/30, \ldots$
(c) $1/2, 1/12, 1/30, 1/56, 1/90, \ldots$ (d) $1/5^3, 3/5^5, 5/5^7, 7/5^9, 9/5^{11}, \ldots$
(e) $1/2!, -1/4!, 1/6!, -1/8!, 1/10!, \ldots$

Ans. (a) $\frac{n}{n+1}$; (b) $(-1)^{n-1}\frac{1}{n^2+n}$; (c) $\frac{1}{(2n-1)2n}$; (d) $\frac{2n-1}{5^{2n+1}}$; (e) $(-1)^{n-1}\frac{1}{(2n)!}$

19. Evaluate:

(a) $\lim_{x\to 2}(x^2-4x)$ (b) $\lim_{x\to -1}(x^3+2x^2-3x-4)$ (c) $\lim_{x\to 1}\frac{(3x-1)^2}{(x+1)^3}$

(d) $\lim_{x\to 0}\frac{3^x-3^{-x}}{3^x+3^{-x}}$ (e) $\lim_{x\to 2}\frac{x-1}{x^2-1}$ (f) $\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}$

(g) $\lim_{x\to -1}\frac{x^2+3x+2}{x^2+4x+3}$ (h) $\lim_{x\to 2}\frac{x-2}{x^2-4}$ (i) $\lim_{x\to 2}\frac{x-2}{\sqrt{x^2-4}}$

(j) $\lim_{x\to 2}\frac{\sqrt{x-2}}{x^2-4}$ (k) $\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$ (l) $\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}$

Ans. (a) -4; (b) 0; (c) $\frac{1}{2}$; (d) 0; (e) $\frac{1}{3}$; (f) -4; (g) $\frac{1}{2}$; (h) $\frac{1}{4}$; (i) 0; (j) ∞, no limit; (k) $3x^2$; (l) 2

20. Evaluate:

(a) $\lim_{x\to\infty}\frac{2x+3}{4x-5}$ (b) $\lim_{x\to\infty}\frac{2x^2+1}{6+x-3x^2}$ (c) $\lim_{x\to\infty}\frac{x}{x^2+5}$ (d) $\lim_{x\to\infty}\frac{x^2+5x+6}{x+1}$

(e) $\lim_{x\to\infty}\frac{x+3}{x^2+5x+6}$ (f) $\lim_{x\to+\infty}\frac{3^x-3^{-x}}{3^x+3^{-x}}$ (g) $\lim_{x\to-\infty}\frac{3^x-3^{-x}}{3^x+3^{-x}}$

Ans. (a) $\frac{1}{2}$; (b) $-\frac{2}{3}$; (c) 0; (d) ∞, no limit; (e) 0; (f) 1; (g) -1

21. Find $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ for the functions f in Problems 11, 12, 13, 15, 16(a), (b), (d), and (g), and 18(b), (c), (g), and (i) of Chapter 6.

Ans. 11. $2a-4$; 12. $\frac{2}{(a+1)^2}$; 13. $2a-1$; 15. $-\frac{27}{(4a-5)^2}$; 16. (a) $2a$, (b) $\frac{a}{\sqrt{a^2+4}}$, (d) $\frac{3}{(a+3)^2}$, (g) $\frac{4a}{(a^2+1)^2}$; 18. (a) $-2a$, (b) 1, (c) no limit, (g) -1, (i) no limit

22. What is $\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{b_0x^n+b_1x^{n-1}+\cdots+b_n}$, where $a_0b_0\neq 0$ and m and n are positive integers, when (a) $m>n$; (b) $m=n$; (c) $m<n$? *Ans.* (a) no limit; (b) a_0/b_0; (c) 0

23. Investigate the behavior of $f(x) = |x|$ as $x \to 0$. Draw a graph. (*Hint*: Examine $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.)

Ans. $\lim_{x\to 0} |x| = 0$

24. Investigate the behavior of $\begin{cases} f(x) = x & x > 0 \\ f(x) = x + 1 & x \le 0 \end{cases}$ as $x \to 0$. Draw a graph.

Ans. $\lim_{x\to 0} f(x)$ does not exist.

25. (*a*) Use Theorem 7.4 and mathematical induction to prove $\lim_{x\to a} x^n = a^n$, for n a positive integer.
(*b*) Use Theorem 7.3 and mathematical induction to prove

$$\lim_{x\to a} [f_1(x) + f_2(x) + \cdots + f_n(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + \cdots + \lim_{x\to a} f_n(x)$$

26. Use Theorem 7.2 and the results of Problem 25 to prove $\lim_{x\to a} P(x) = P(a)$, where $P(x)$ is any polynomial in x.

27. For $f(x) = 5x - 6$, find a $\delta > 0$ such that whenever $0 < |x - 4| < \delta$, then $|f(x) - 14| < \epsilon$, when (*a*) $\epsilon = \frac{1}{2}$ and (*b*) $\epsilon = 0.001$. *Ans.* (*a*) $\frac{1}{10}$; (*b*) 0.0002

28. Use the precise definition to prove (*a*) $\lim_{x\to 3} 5x = 15$; (*b*) $\lim_{x\to 2} x^2 = 4$; (*c*) $\lim_{x\to 2} (x^2 - 3x + 5) = 3$.

29. Use the precise definition to prove

(*a*) $\lim_{x\to 0} \frac{1}{x} = \infty$ (*b*) $\lim_{x\to 1} \frac{x}{x-1} = \infty$ (*c*) $\lim_{x\to\infty} \frac{x}{x-1} = 1$ (*d*) $\lim_{x\to\infty} \frac{x^2}{x+1} = \infty$

30. Prove: If $f(x)$ is defined for all x near $x = a$ and has a limit as $x \to a$, that limit is unique. (*Hint*: Assume $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} f(x) = B$, and $B \ne A$. Choose $\epsilon_1, \epsilon_2 < \frac{1}{2}|A - B|$. Determine δ_1 and δ_2 for the two limits and take δ the smaller of δ_1 and δ_2. Show that then $|A - B| = |[A - f(x)] + [f(x) - B]| < |A - B|$, a contradiction.)

31. Let $f(x)$, $g(x)$, and $h(x)$ be such that (1) $f(x) \le g(x) \le h(x)$ for all values of x near $x = a$ and (2) $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = A$. Show that $\lim_{x\to a} g(x) = A$. (*Hint*: For a given $\epsilon > 0$, however small, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - A| < \epsilon$ and $|h(x) - A| < \epsilon$ or $A - \epsilon < f(x) \le g(x) \le h(x) < A + \epsilon$.)

32. Prove: If $f(x) \le M$ for all x and if $\lim_{x\to a} f(x) = A$, then $A \le M$. (*Hint*: Suppose $A > M$. Choose $\epsilon = \frac{1}{2}(A - M)$ and obtain a contradiction.)

Chapter 8

Continuity

A FUNCTION $f(x)$ IS CONTINUOUS at $x = x_0$ if

$$f(x_0) \text{ is defined} \qquad \lim_{x \to x_0} f(x) \text{ exists} \qquad \lim_{x \to x_0} f(x) = f(x_0)$$

For example, $f(x) = x^2 + 1$ is continuous at $x = 2$ since $\lim_{x \to 2} f(x) = 5 = f(2)$. The first condition above implies that a function can be continuous only at points of its domain. Thus, $f(x) = \sqrt{4 - x^2}$ is not continuous at $x = 3$ because $f(3)$ is imaginary, i.e., is not defined.

A function $f(x)$ is called *continuous* if it is continuous at every point of its domain. Thus, $f(x) = x^2 + 1$ and all other polynomials in x are continuous functions; other examples are e^x, $\sin x$, and $\cos x$.

A function f is said to be *continuous on a closed interval* $[a, b]$ if the function that restricts f to $[a, b]$ is continuous at each point of $[a, b]$; in other words, we ignore what happens to the left of a and to the right of b. Consider, for example, the function f such that $f(x) = x$ for $0 \le x \le 1$, $f(x) = -1$ for $x < 0$, and $f(x) = 2$ for $x > 1$. This function is continuous at every point except $x = 0$ and $x = 1$. However, the function is continuous on the interval $[0, 1]$ because, for that interval, we are considering the function g whose domain is $[0, 1]$ such that $g(x) = x$ for x in $[0, 1]$. Because

$$\lim_{x \to 0} g(x) = \lim_{x \to 0^+} g(x) = 0 \qquad \text{and} \qquad \lim_{x \to 1} g(x) = \lim_{x \to 1^-} g(x) = 1$$

g is continuous at 0 and 1 (and, clearly, at all points between 0 and 1).

A FUNCTION $f(x)$ IS DISCONTINUOUS at $x = x_0$ if one or more of the conditions for continuity fails there.

EXAMPLE 1: (*a*) $f(x) = \dfrac{1}{x - 2}$ is discontinuous at $x = 2$ because $f(2)$ is not defined (has zero as denominator) and because $\lim_{x \to 2} f(x)$ does not exist (equals ∞). The function is, however, continuous everywhere except at $x = 2$, where it is said to have an *infinite discontinuity*. See Fig. 8-1.

(*b*) $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2$ because $f(2)$ is not defined (both numerator and denominator are zero) and because $\lim_{x \to 2} f(x) = 4$. The discontinuity here is called *removable* since it may be removed by redefining the function as $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x \ne 2$; $f(2) = 4$. (Note that the discontinuity in (*a*) cannot be so removed because the limit also does not exist.) The graphs of $f(x) = \dfrac{x^2 - 4}{x - 2}$ and $g(x) = x + 2$ are identical except at $x = 2$, where the former has a 'hole' (see Fig. 8-2). Removing the discontinuity consists simply of filling the 'hole.'

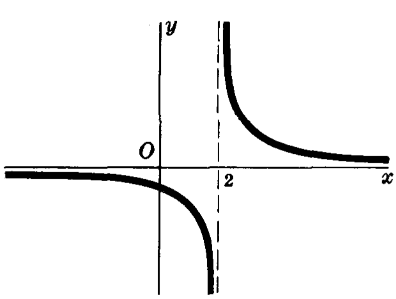


Fig. 8-1

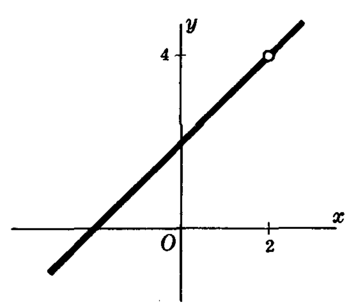


Fig. 8-2

(*c*) $f(x) = \dfrac{x^3 - 27}{x - 3}$ for $x \neq 3$; $f(3) = 9$ is discontinuous at $x = 3$ because $f(3) = 9$ while $\lim_{x\to 3} f(x) = 27$, so that $\lim_{x\to 3} f(x) \neq f(3)$. The discontinuity may be removed by redefining the function as $f(x) = \dfrac{x^3 - 27}{x - 3}$ for $x \neq 3$; $f(3) = 27$.

(*d*) The function of Problem 4 of Chapter 6 is defined for all $x > 0$ but has discontinuities at $x = 1, 2, 3, \ldots$ (see Problem 13 of Chapter 7) arising from the fact that

$$\lim_{x\to s^-} f(x) \neq \lim_{x\to s^+} f(x) \qquad \text{for } s \text{ any positive integer}$$

These are called *jump discontinuities*. (See Problems 1 and 2.)

PROPERTIES OF CONTINUOUS FUNCTIONS. The theorems on limits in Chapter 7 lead readily to theorems on continuous functions. In particular, if $f(x)$ and $g(x)$ are continuous at $x = a$, so also are $f(x) \pm g(x)$, $f(x)g(x)$, and $f(x)/g(x)$, provided in the latter that $g(a) \neq 0$. Hence, polynomials in x are everywhere continuous whereas rational functions of x are continuous everywhere except at values of x for which the denominator is zero.

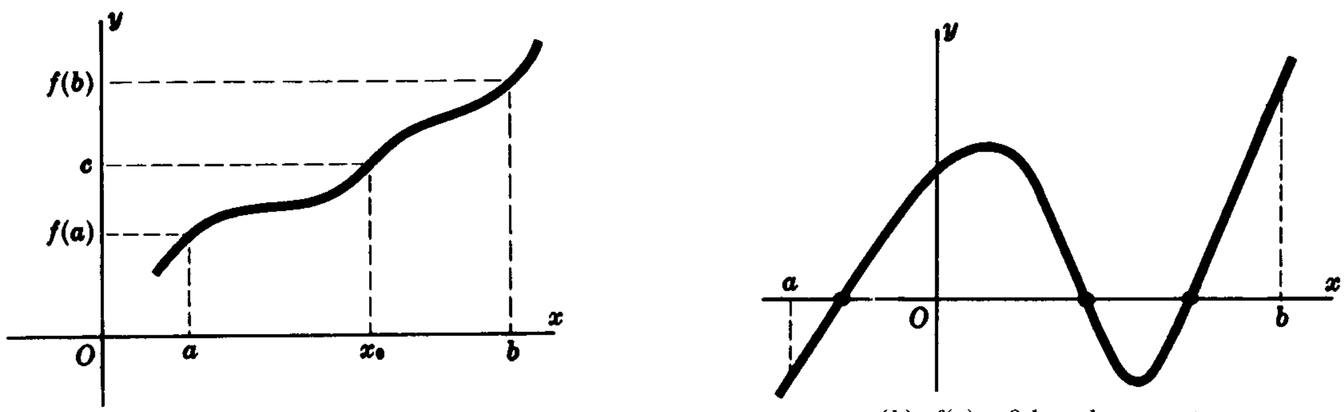
You have probably used certain properties of continuous functions in your study of algebra:

1. In sketching the graph of a polynomial $y = f(x)$, any two points $(a, f(a))$ and $(b, f(b))$ are joined by an unbroken arc.
2. If $f(a)$ and $f(b)$ have opposite signs, the graph of $y = f(x)$ crosses the x axis at least once, and the equation $f(x) = 0$ has at least one root between $x = a$ and $x = b$.

The property of continuous functions used here is

Property 8.1: If $f(x)$ is continuous on the interval $a \le x \le b$ and if $f(a) \neq f(b)$, then for any number c between $f(a)$ and $f(b)$ there is at least one value of x, say $x = x_0$, for which $f(x_0) = c$ and $a \le x_0 \le b$.

Figure 8-3 illustrates the two applications of this property, and Fig. 8-4 shows that continuity throughout the interval is essential.



(*a*)

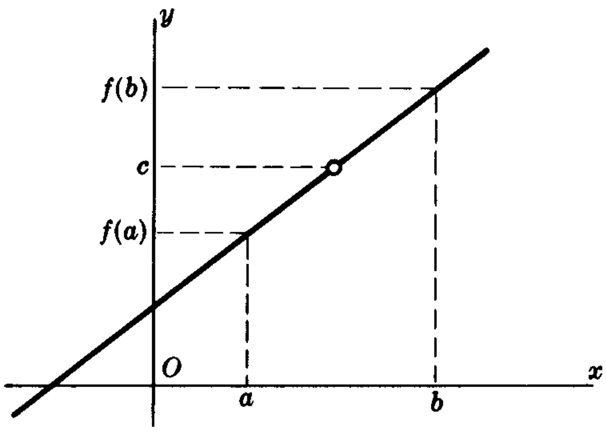
(*b*) $f(x) = 0$ has three roots between $x = a$ and $x = b$.

Fig. 8-3

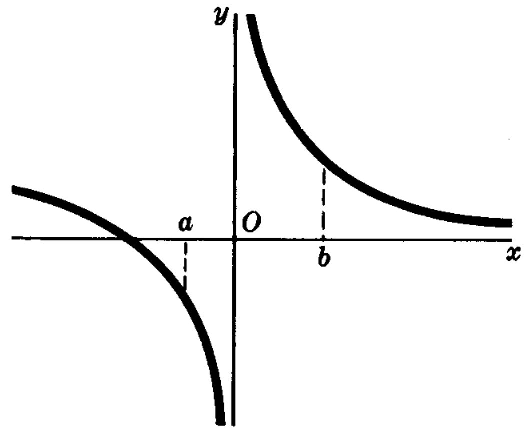
Other properties of continuous functions are important here:

Property 8.2: If $f(x)$ is continuous on the interval $a \le x \le b$, then $f(x)$ takes on a least value m and a greatest value M on the interval.

Although a proof of Property 8.2 is beyond the scope of this book, the property will be used freely in later chapters. Consider Figure 8-5(*a*)–(*c*). In Fig. 8-5(*a*) the function is continuous on $a \le x \le b$; the least value m and the greatest value M occur at $x = c$ and $x = d$ respectively, both points being within the interval. In Fig. 8-5(*b*) the function is continuous on $a \le x \le b$; the least value occurs at the endpoint $x = a$, while the greatest value occurs at $x = c$ within the interval. In Fig. 8-5(*c*) there is a discontinuity at $x = c$, where $a < c < b$; the function has a least value at $x = a$ but no greatest value.

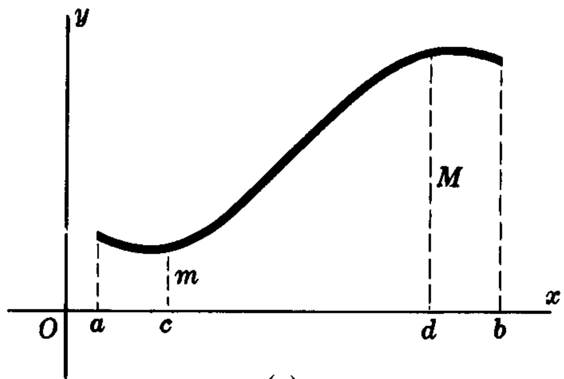


(a)

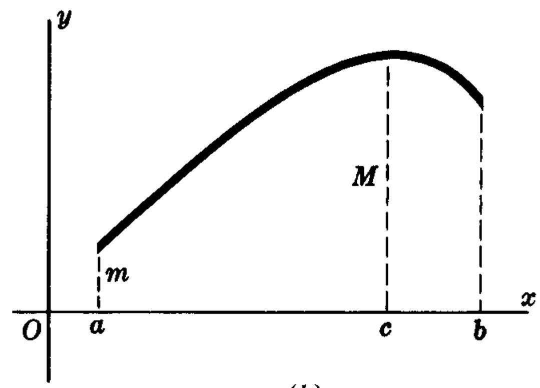


(b) $f(x) = 0$ has no root between $x = a$ and $x = b$.

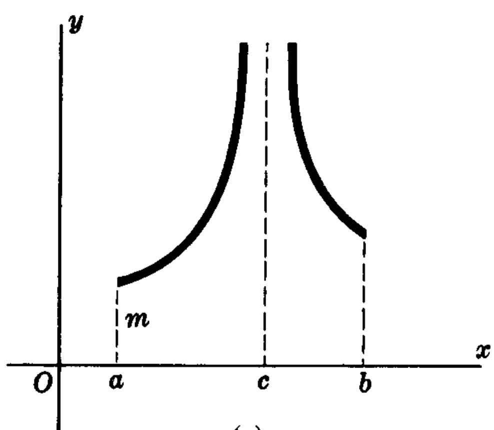
Fig. 8-4



(a)



(b)



(c)

Fig. 8-5

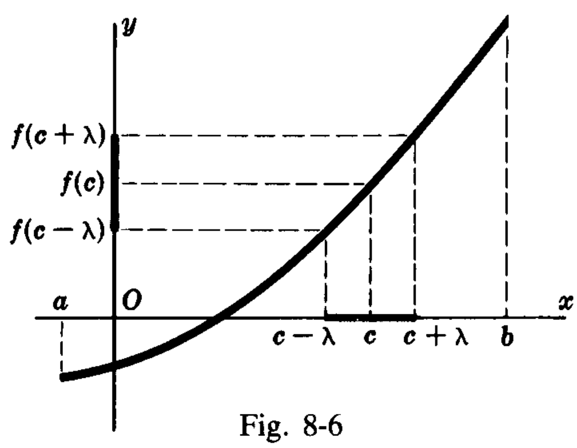


Fig. 8-6

Property 8.3: If $f(x)$ is continuous on the interval $a \le x \le b$, and if c is any number between a and b and $f(c) > 0$, then there exists a number $\lambda > 0$ such that whenever $c - \lambda < x < c + \lambda$, then $f(x) > 0$.

This property is illustrated in Fig. 8-6. For a proof, see Problem 4.

Solved Problems

1. Use Problem 10 of Chapter 7 to find the discontinuities of:

(a) $f(x) = 2/x$: Has an infinite discontinuity at $x = 0$.

(b) $f(x) = \dfrac{x-1}{(x+3)(x-2)}$: Has infinite discontinuities at $x = -3$ and $x = 2$.

(c) $f(x) = \dfrac{(x+2)(x-1)}{(x-3)^2}$: Has an infinite discontinuity at $x = 3$.

2. Use Problem 6 of Chapter 7 to find the discontinuities of:

(a) $f(x) = \dfrac{x^2-27}{x^2-9}$: Has a removable discontinuity at $x = 3$. There is also an infinite discontinuity at $x = -3$.

(b) $f(x) = \dfrac{4-x^3}{3-\sqrt{x^2+5}}$: Has a removable discontinuity at $x = 2$. There is also a removable discontinuity at $x = -2$.

(c) $f(x) = \dfrac{x^2+x-2}{(x-1)^2}$: Has an infinite discontinuity at $x = 1$.

3. Show that the existence of $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ implies $f(x)$ is continuous at $x = a$.

The existence of the limit implies that $f(a+h) - f(a) \to 0$ as $h \to 0$. Thus, $\lim\limits_{h\to 0} f(a+h) = f(a)$ and $f(x)$ is continuous at $x = a$.

4. Prove: If $f(x)$ is continuous on the interval $a \le x \le b$, and if c is any number between a and b and $f(c) > 0$, then there exists a number $\lambda > 0$ such that whenever $c - \lambda < x < c + \lambda$, then $f(x) > 0$.

Since $f(x)$ is continuous at $x = c$, $\lim\limits_{x\to c} f(x) = f(c)$ and for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{Whenever } 0 < |x - c| < \delta \text{ then } |f(x) - f(c)| < \epsilon \tag{1}$$

Now $f(x) > 0$ at all points on the interval $c - \delta < x < c + \delta$ for which $f(x) \ge f(c)$. At all other points of the interval $f(x) < f(c)$ so that $|f(x) - f(c)| = f(c) - f(x) < \epsilon$ and $f(x) > f(c) - \epsilon$. Thus, at these points, $f(x) > 0$ unless $\epsilon \ge f(c)$. Hence, to determine an interval meeting the requirements of the theorem, select $\epsilon < f(c)$, determine δ satisfying (*1*), and take $\lambda < \delta$. (See Problem 10 for the companion theorem.)

Supplementary Problems

5. Examine the functions of Problem 19(*a*) to (*h*) of Chapter 7 for points of discontinuity.

Ans. (*a*), (*b*), (*d*) none; (*c*) $x = -1$; (*e*) $x = \pm 1$; (*f*) $x = 2, 3$; (*g*) $x = -1, -3$; (*h*) $x = \pm 2$

6. Show that $f(x) = |x|$ is everywhere continuous.

7. Show that $f(x) = \dfrac{1 - 2^{1/x}}{1 + 2^{1/x}}$ has a jump discontinuity at $x = 0$.

8. Show that at $x = 0$, (*a*) $f(x) = \dfrac{1}{3^{1/x} + 1}$ has a jump discontinuity and (*b*) $f(x) = \dfrac{x}{3^{1/x} + 1}$ has a removable discontinuity.

9. If Fig. 8-4(*a*) is the graph of $f(x) = \dfrac{x^2 - 4x - 21}{x - 7}$, show that there is a removable discontinuity at $x = 7$ and that $c = 10$ there.

10. Prove: If $f(x)$ is continuous on the interval $a \le x \le b$, and if c is any number between a and b and $f(c) < 0$, then there exists a number $\lambda > 0$ such that whenever $c - \lambda < x < c + \lambda$ then $f(x) < 0$.

11. Sketch the graph of each of the following functions, find any discontinuities, and state why the function fails to be continuous at those points. Indicate which discontinuities are removable.

(*a*) $f(x) = \dfrac{|x|}{x}$ (*b*) $f(x) = \dfrac{x^2 - 3x - 10}{x + 2}$ (*c*) $f(x) = \begin{cases} x + 3 & \text{if } x \ge 2 \\ x^2 + 1 & \text{if } x < 2 \end{cases}$

(*d*) $f(x) = |x| - x$ (*e*) $f(x) = \begin{cases} 4 - x & \text{if } x \ge 3 \\ x - 2 & \text{if } 0 < x < 3 \\ x - 1 & \text{if } x \le 0 \end{cases}$ (*f*) $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$

(*g*) $f(x) = \dfrac{x^3 + x^2 - 17x + 15}{x^2 + 2x - 15}$

Ans. (*a*) $x = 0$; (*b*) $x = -2$ (removable); (*c*), (*d*) no discontinuities; (*e*) $x = 0$; (*f*) $x = 1, -1$ (both removable); (*g*) $x = 3, -5$ (both removable)

12. Sketch the graphs of the following functions, and determine whether they are continuous on the closed interval $[0, 1]$.

(*a*) $f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x > 1 \end{cases}$ (*b*) $f(x) = \begin{cases} \dfrac{1}{x} & \text{for } x > 0 \\ 1 & \text{for } x \le 0 \end{cases}$ (*c*) $f(x) = \begin{cases} -1 & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$

(*d*) $f(x) = 1$ for $0 < x \le 1$ (*e*) $f(x) = \begin{cases} x & \text{for } x \le 0 \\ 0 & \text{for } 0 < x < 1 \\ x & \text{for } x \ge 1 \end{cases}$

Chapter 9

The Derivative

INCREMENTS. The *increment* Δx of a variable x is the change in x as it increases or decreases from one value $x = x_0$ to another value $x = x_1$ in its domain. Here, $\Delta x = x_1 - x_0$ and we may write $x_1 = x_0 + \Delta x$.

If the variable x is given an increment Δx from $x = x_0$ (that is, if x changes from $x = x_0$ to $x = x_0 + \Delta x$) and a function $y = f(x)$ is thereby given an increment $\Delta y = f(x_0 + \Delta x) - f(x_0)$ from $y = f(x_0)$, then the quotient

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$$

is called the *average rate of change* of the function on the interval between $x = x_0$ and $x = x_0 + \Delta x$.

EXAMPLE 1: When x is given the increment $\Delta x = 0.5$ from $x_0 = 1$, the function $y = f(x) = x^2 + 2x$ is given the increment $\Delta y = f(1 + 0.5) - f(1) = 5.25 - 3 = 2.25$. Thus, the average rate of change of y on the interval between $x = 1$ and $x = 1.5$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{2.25}{0.5} = 4.5$.

(See Problems 1 and 2.)

THE DERIVATIVE of a function $y = f(x)$ with respect to x at the point $x = x_0$ is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

provided the limit exists. This limit is also called the *instantaneous rate of change* (or simply, the *rate of change*) of y with respect to x at $x = x_0$.

EXAMPLE 2: Find the derivative of $y = f(x) = x^2 + 3x$ with respect to x at $x = x_0$. Use this to find the value of the derivative at (a) $x_0 = 2$ and (b) $x_0 = -4$.

$$\begin{aligned}
y_0 &= f(x_0) = x_0^2 + 3x_0 \\
y_0 + \Delta y &= f(x_0 + \Delta x) = (x_0 + \Delta x)^2 + 3(x_0 + \Delta x) \\
&= x_0^2 + 2x_0\,\Delta x + (\Delta x)^2 + 3x_0 + 3\,\Delta x \\
\Delta y &= f(x_0 + \Delta x) - f(x_0) = 2x_0\,\Delta x + 3\,\Delta x + (\Delta x)^2 \\
\frac{\Delta y}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 2x_0 + 3 + \Delta x
\end{aligned}$$

The derivative at $x = x_0$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + 3 + \Delta x) = 2x_0 + 3$$

(a) At $x_0 = 2$, the value of the derivative is $2(2) + 3 = 7$.
(b) At $x_0 = -4$, the value of the derivative is $2(-4) + 3 = -5$.

IN FINDING DERIVATIVES it is customary to drop the subscript 0 and obtain the derivative of $y = f(x)$ *with respect to* x as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The derivative of $y = f(x)$ with respect to x may be indicated by any one of the symbols

$$\frac{d}{dx}y \qquad \frac{dy}{dx} \qquad D_x y \qquad y' \qquad f'(x) \qquad \frac{d}{dx}f(x)$$

(See Problems 3 to 8.)

DIFFERENTIABILITY. A function is said to be *differentiable* at a point $x = x_0$ if the derivative of the function exists at that point. Problem 3 of Chapter 8 shows that differentiability implies continuity. The converse is false (see Problem 11).

Solved Problems

1. Given $y = f(x) = x^2 + 5x - 8$, find Δy and $\Delta y/\Delta x$ as x changes (*a*) from $x_0 = 1$ to $x_1 = x_0 + \Delta x = 1.2$ and (*b*) from $x_0 = 1$ to $x_1 = 0.8$.

(*a*) $\Delta x = x_1 - x_0 = 1.2 - 1 = 0.2$ and
$\Delta y = f(x_0 + \Delta x) - f(x_0) = f(1.2) - f(1) = -0.56 - (-2) = 1.44$. So $\dfrac{\Delta y}{\Delta x} = \dfrac{1.44}{0.2} = 7.2$

(*b*) $\Delta x = 0.8 - 1 = -0.2$ and
$\Delta y = f(0.8) - f(1) = -3.36 - (-2) = -1.36$. So $\dfrac{\Delta y}{\Delta x} = \dfrac{-1.36}{-0.2} = 6.8$

Geometrically, $\Delta y/\Delta x$ in (*a*) is the slope of the secant line joining the points $(1, -2)$ and $(1.2, -0.56)$ of the parabola $y = x^2 + 5x - 8$, and in (*b*) is the slope of the secant line joining the points $(0.8, -3.36)$ and $(1, -2)$ of the same parabola.

2. When a body freely falls a distance s feet from rest in t seconds, $s = 16t^2$. Find $\Delta s/\Delta t$ as t changes from t_0 to $t_0 + \Delta t$. Use this to find $\Delta s/\Delta t$ as t changes (*a*) from 3 to 3.5, (*b*) from 3 to 3.2, and (*c*) from 3 to 3.1.

$$\frac{\Delta s}{\Delta t} = \frac{16(t_0 + \Delta t)^2 - 16t_0^2}{\Delta t} = \frac{32t_0\,\Delta t + 16(\Delta t)^2}{\Delta t} = 32t_0 + 16\,\Delta t$$

(*a*) Here $t_0 = 3$, $\Delta t = 0.5$, and $\Delta s/\Delta t = 32(3) + 16(0.5) = 104$ ft/s.
(*b*) Here $t_0 = 3$, $\Delta t = 0.2$, and $\Delta s/\Delta t = 32(3) + 16(0.2) = 99.2$ ft/s.
(*c*) Here $t_0 = 3$, $\Delta t = 0.1$, and $\Delta s/\Delta t = 97.6$ ft/s.

Since Δs is the displacement of the body from time $t = t_0$ to $t = t_0 + \Delta t$,

$$\frac{\Delta s}{\Delta t} = \frac{\text{displacement}}{\text{time}} = \text{average velocity of the body over the time interval}$$

3. Find dy/dx, given $y = x^3 - x^2 - 4$. Find also the value of dy/dx when (*a*) $x = 4$, (*b*) $x = 0$, (*c*) $x = -1$.

$$\begin{aligned}
y + \Delta y &= (x + \Delta x)^3 - (x + \Delta x)^2 - 4 \\
&= x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^2 - 2x(\Delta x) - (\Delta x)^2 - 4 \\
\Delta y &= (3x^2 - 2x)\,\Delta x + (3x - 1)(\Delta x)^2 + (\Delta x)^3 \\
\frac{\Delta y}{\Delta x} &= 3x^2 - 2x + (3x - 1)\,\Delta x + (\Delta x)^2 \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0} [3x^2 - 2x + (3x - 1)\,\Delta x + (\Delta x)^2] = 3x^2 - 2x
\end{aligned}$$

(*a*) $\left.\dfrac{dy}{dx}\right|_{x=4} = 3(4)^2 - 2(4) = 40$; (*b*) $\left.\dfrac{dy}{dx}\right|_{x=0} = 3(0)^2 - 2(0) = 0$; (*c*) $\left.\dfrac{dy}{dx}\right|_{x=-1} = 3(-1)^2 - 2(-1) = 5$

4. Find the derivative of $y = x^2 + 3x + 5$.

$$y + \Delta y = (x + \Delta x)^2 + 3(x + \Delta x) + 5 = x^2 + 2x\,\Delta x + \Delta x^2 + 3x + 3\,\Delta x + 5$$
$$\Delta y = (2x + 3)\,\Delta x + \Delta x^2$$
$$\frac{\Delta y}{\Delta x} = \frac{(2x + 3)\,\Delta x + \Delta x^2}{\Delta x} = 2x + 3 + \Delta x$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} (2x + 3 + \Delta x) = 2x + 3$$

5. Find the derivative of $y = \dfrac{1}{x - 2}$ at $x = 1$ and $x = 3$.

$$y + \Delta y = \frac{1}{x + \Delta x - 2}$$
$$\Delta y = \frac{1}{x + \Delta x - 2} - \frac{1}{x - 2} = \frac{(x - 2) - (x + \Delta x - 2)}{(x - 2)(x + \Delta x - 2)} = \frac{-\Delta x}{(x - 2)(x + \Delta x - 2)}$$
$$\frac{\Delta y}{\Delta x} = \frac{-1}{(x - 2)(x + \Delta x - 2)}$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-1}{(x - 2)(x + \Delta x - 2)} = \frac{-1}{(x - 2)^2}$$

At $x = 1$, $\dfrac{dy}{dx} = \dfrac{-1}{(1 - 2)^2} = -1$; at $x = 3$, $\dfrac{dy}{dx} = \dfrac{-1}{(3 - 2)^2} = -1$.

6. Find the derivative of $f(x) = \dfrac{2x - 3}{3x + 4}$.

$$f(x + \Delta x) = \frac{2(x + \Delta x) - 3}{3(x + \Delta x) + 4}$$
$$f(x + \Delta x) - f(x) = \frac{2x + 2\,\Delta x - 3}{3x + 3\,\Delta x + 4} - \frac{2x - 3}{3x + 4}$$
$$= \frac{(3x + 4)[(2x - 3) + 2\,\Delta x] - (2x - 3)[(3x + 4) + 3\,\Delta x]}{(3x + 4)(3x + 3\,\Delta x + 4)}$$
$$= \frac{(6x + 8 - 6x + 9)\,\Delta x}{(3x + 4)(3x + 3\,\Delta x + 4)} = \frac{17\,\Delta x}{(3x + 4)(3x + 3\,\Delta x + 4)}$$
$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{17}{(3x + 4)(3x + 3\,\Delta x + 4)}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{17}{(3x + 4)(3x + 3\,\Delta x + 4)} = \frac{17}{(3x + 4)^2}$$

7. Find the derivative of $y = \sqrt{2x + 1}$.

$$y + \Delta y = (2x + 2\,\Delta x + 1)^{1/2}$$
$$\Delta y = (2x + 2\,\Delta x + 1)^{1/2} - (2x + 1)^{1/2}$$
$$= [(2x + 2\,\Delta x + 1)^{1/2} - (2x + 1)^{1/2}]\,\frac{(2x + 2\,\Delta x + 1)^{1/2} + (2x + 1)^{1/2}}{(2x + 2\,\Delta x + 1)^{1/2} + (2x + 1)^{1/2}}$$
$$= \frac{(2x + 2\,\Delta x + 1) - (2x + 1)}{(2x + 2\,\Delta x + 1)^{1/2} + (2x + 1)^{1/2}} = \frac{2\,\Delta x}{(2x + 2\,\Delta x + 1)^{1/2} + (2x + 1)^{1/2}}$$
$$\frac{\Delta y}{\Delta x} = \frac{2}{(2x + 2\,\Delta x + 1)^{1/2} + (2x + 1)^{1/2}}$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2}{(2x + 2\,\Delta x + 1)^{1/2} + (2x + 1)^{1/2}} = \frac{1}{(2x + 1)^{1/2}}$$

For the function $f(x) = \sqrt{2x+1}$, $\lim_{x\to(-1/2)^+} f(x) = 0 = f(-\frac{1}{2})$ while $\lim_{x\to(-1/2)^-} f(x)$ does not exist; the function has *right-hand continuity* at $x = -\frac{1}{2}$. At $x = -\frac{1}{2}$, the derivative is infinite.

8. Find the derivative of $f(x) = x^{1/3}$. Examine $f'(0)$.

$$f(x+\Delta x) = (x+\Delta x)^{1/3}$$

$$f(x+\Delta x) - f(x) = (x+\Delta x)^{1/3} - x^{1/3}$$

$$= \frac{[(x+\Delta x)^{1/3} - x^{1/3}][(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}]}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$= \frac{x+\Delta x - x}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}}$$

The derivative does not exist at $x = 0$ because the denominator is zero there. However, the function is continuous at $x = 0$. This, together with the remark at the end of Problem 7, illustrates: *If the derivative of a function exists at $x = a$ then the function is continuous there, but not conversely.*

9. Interpret dy/dx geometrically.

From Fig. 9-1 we see that $\Delta y/\Delta x$ is the slope of the secant line joining an arbitrary but fixed point $P(x, y)$ and a nearby point $Q(x+\Delta x, y+\Delta y)$ of the curve. As $\Delta x \to 0$, P remains fixed while Q moves along the curve toward P, and the line PQ revolves about P toward its limiting position, the tangent line PT to the curve at P. Thus, dy/dx gives the slope of the tangent at P to the curve $y = f(x)$.

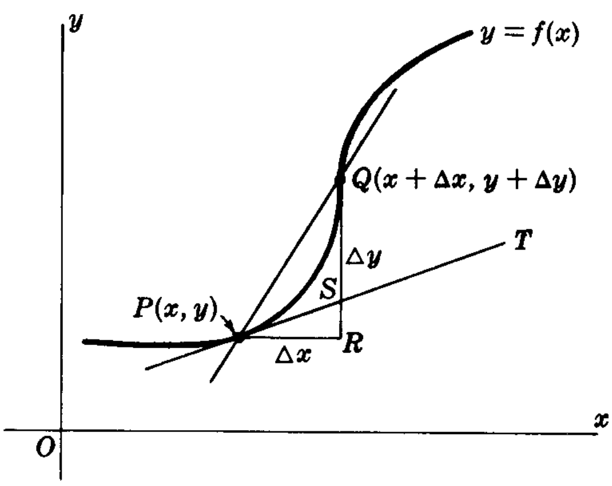


Fig. 9-1

For example, from Problem 3, the slope of the cubic $y = x^3 - x^2 - 4$ is $m = 40$ at the point $x = 4$; it is $m = 0$ at the point $x = 0$; and it is $m = 5$ at the point $x = -1$.

10. Find ds/dt for the function of Problem 2 and interpret it physically.

Here

$$\frac{\Delta s}{\Delta t} = 32t_0 + 16\,\Delta t \qquad \text{and} \qquad \frac{ds}{dt} = \lim_{\Delta t\to 0}(32t_0 + 16\,\Delta t) = 32t_0$$

As $\Delta t \to 0$, $\Delta s/\Delta t$ gives the average velocity of the body for shorter and shorter time intervals Δt. Then we can define ds/dt to be the instantaneous velocity v of the body at time $t = t_0$. For example, at $t = 3$, $v = 32(3) = 96$ ft/s.

11. Find $f'(x)$, given $f(x) = |x|$.

The function is continuous for all values of x. For $x < 0$, $f(x) = -x$ and $f'(x) = \lim_{\Delta x \to 0} \dfrac{-(x + \Delta x) - (-x)}{\Delta x} = -1$; for $x > 0$, $f(x) = x$ and $f'(x) = 1$.

At $x = 0$, $f(x) = 0$ and $\lim_{\Delta x \to 0} \dfrac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{|\Delta x|}{\Delta x}$. As $\Delta x \to 0^-$, $\dfrac{|\Delta x|}{\Delta x} \to -1$; but as $\Delta x \to 0^+$, $\dfrac{|\Delta x|}{\Delta x} \to 1$. Hence, the derivative does not exist at $x = 0$.

12. Compute $\epsilon = \dfrac{\Delta y}{\Delta x} - \dfrac{dy}{dx}$ for the function of (*a*) Problem 3 and (*b*) Problem 5. Verify that $\epsilon \to 0$ as $\Delta x \to 0$.

(*a*) $\epsilon = [3x^2 - 2x + (3x - 1)\,\Delta x + (\Delta x)^2] - (3x^2 - 2x) = (3x - 1 + \Delta x)\,\Delta x$

(*b*) $\epsilon = \dfrac{-1}{(x-2)(x + \Delta x - 2)} - \dfrac{-1}{(x-2)^2} = \dfrac{-(x-2) + (x + \Delta x - 2)}{(x-2)^2(x + \Delta x - 2)} = \dfrac{1}{(x-2)^2(x + \Delta x - 2)}\,\Delta x$

Both obviously go to zero as $\Delta x \to 0$.

13. Interpret $\Delta y = \dfrac{dy}{dx}\,\Delta x + \epsilon\,\Delta x$ of Problem 12 geometrically.

In Fig. 9-1, $\Delta y = RQ$ and $\dfrac{dy}{dx}\,\Delta x = PR \tan \angle TPR = RS$; thus, $\epsilon\,\Delta x = SQ$. For a change Δx in x from $P(x, y)$, Δy is the corresponding change in y *along the curve* while $\dfrac{dy}{dx}\,\Delta x$ is the corresponding change in y *along the tangent line PT*. Since their difference $\epsilon\,\Delta x$ is a multiple of $(\Delta x)^2$, it goes to zero faster than Δx, and $\dfrac{dy}{dx}\,\Delta x$ can be used as an approximation of Δy when $|\Delta x|$ is small.

Supplementary Problems

14. Find Δy and $\Delta y/\Delta x$, given
(*a*) $y = 2x - 3$ and x changes from 3.3 to 3.5.
(*b*) $y = x^2 + 4x$ and x changes from 0.7 to 0.85.
(*c*) $y = 2/x$ and x changes from 0.75 to 0.5.

Ans. (*a*) 0.4 and 2; (*b*) 0.8325 and 5.55; (*c*) $\frac{4}{3}$ and $-\frac{16}{3}$

15. Find Δy, given $y = x^2 - 3x + 5$, $x = 5$, and $\Delta x = -0.01$. What then is the value of y when $x = 4.99$?

Ans. $\Delta y = -0.0699$; $y = 14.9301$

16. Find the average velocity, given
(*a*) $s = (3t^2 + 5)$ ft and t changes from 2 to 3 s.
(*b*) $s = (2t^2 + 5t - 3)$ ft and t changes from 2 to 5 s.

Ans. (*a*) 15 ft/s; (*b*) 19 ft/s

17. Find the increase in the volume of a spherical balloon when its radius is increased (*a*) from r to $r + \Delta r$ in; (*b*) from 2 to 3 in. *Ans.* (*a*) $\frac{4}{3}\pi(3r^2 + 3r\,\Delta r + \Delta r)^2\,\Delta r$ in^3; (*b*) $\frac{76}{3}\pi$ in^3

18. Find the derivative of each of the following:

(a) $y = 4x - 3$ (b) $y = 4 - 3x$ (c) $y = x^2 + 2x - 3$
(d) $y = 1/x^2$ (e) $y = (2x - 1)/(2x + 1)$ (f) $y = (1 + 2x)/(1 - 2x)$
(g) $y = \sqrt{x}$ (h) $y = 1/\sqrt{x}$ (i) $y = \sqrt{1 + 2x}$
(j) $y = 1/\sqrt{2 + x}$

Ans. (a) 4; (b) -3; (c) $2(x + 1)$; (d) $-2/x^3$; (e) $\dfrac{4}{(2x + 1)^2}$; (f) $\dfrac{4}{(1 - 2x)^2}$; (g) $\dfrac{1}{2\sqrt{x}}$; (h) $-\dfrac{1}{2x\sqrt{x}}$; (i) $\dfrac{1}{\sqrt{1 + 2x}}$; (j) $-\dfrac{1}{2(2 + x)^{3/2}}$

19. Find the slope of the following curves at the point $x = 1$:

(a) $y = 8 - 5x^2$ (b) $y = \dfrac{4}{x + 1}$ (c) $y = \dfrac{2}{x + 3}$

Ans. (a) -10; (b) -1; (c) $-\frac{1}{8}$

20. Find the coordinates of the vertex v of the parabola $y = x^2 - 4x + 1$ by making use of the fact that at the vertex the slope of the tangent is zero. *Ans.* $V(2, -3)$

21. Find the slope of the tangents to the parabola $y = -x^2 + 5x - 6$ at its points of intersection with the x axis. *Ans.* at $x = 2$, $m = 1$; at $x = 3$, $m = -1$

22. When s is measured in feet and t in seconds, find the velocity at time $t = 2$ of the following motions:

(a) $s = t^2 + 3t$ (b) $s = t^3 - 3t^2$ (c) $s = \sqrt{t + 2}$

Ans. (a) 7 ft/s; (b) 0 ft/s; (c) $\frac{1}{4}$ ft/s

23. Show that the instantaneous rate of change of the volume of a cube with respect to its edge x in inches is 12 in^3/in when $x = 2$ in.

Chapter 10

Rules for Differentiating Functions

DIFFERENTIATION. Recall that a function f is said to be differentiable at $x = x_0$ if the derivative $f'(x_0)$ exists. A function is said to be differentiable on an interval if it is differentiable at every point of the interval. The functions of elementary calculus are differentiable, except possibly at isolated points, on their intervals of definition. The process of finding the derivative of a function is called *differentiation*.

DIFFERENTIATION FORMULAS. In the following formulas u, v, and w are differentiable functions of x, and c and m are constants.

1. $\dfrac{d}{dx}(c) = 0$
2. $\dfrac{d}{dx}(x) = 1$
3. $\dfrac{d}{dx}(u + v + \cdots) = \dfrac{d}{dx}(u) + \dfrac{d}{dx}(v) + \cdots$
4. $\dfrac{d}{dx}(cu) = c\,\dfrac{d}{dx}(u)$
5. $\dfrac{d}{dx}(uv) = u\,\dfrac{d}{dx}(v) + v\,\dfrac{d}{dx}(u)$
6. $\dfrac{d}{dx}(uvw) = uv\,\dfrac{d}{dx}(w) + uw\,\dfrac{d}{dx}(v) + vw\,\dfrac{d}{dx}(u)$
7. $\dfrac{d}{dx}\left(\dfrac{u}{c}\right) = \dfrac{1}{c}\,\dfrac{d}{dx}(u),\ c \neq 0$
8. $\dfrac{d}{dx}\left(\dfrac{c}{u}\right) = c\,\dfrac{d}{dx}\left(\dfrac{1}{u}\right) = -\dfrac{c}{u^2}\,\dfrac{d}{dx}(u),\ u \neq 0$
9. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\,\dfrac{d}{dx}(u) - u\,\dfrac{d}{dx}(v)}{v^2},\ v \neq 0$
10. $\dfrac{d}{dx}(x^m) = mx^{m-1}$
11. $\dfrac{d}{dx}(u^m) = mu^{m-1}\,\dfrac{d}{dx}(u)$

(See Problems 1 to 13.)

INVERSE FUNCTIONS. Two functions f and g such that $g(f(x)) = x$ and $f(g(y)) = y$ are said to be *inverse* functions. Inverse functions reverse the effect of each other.

EXAMPLE 1: (*a*) The inverse of $f(x) = x + 1$ is the function $g(y) = y - 1$.
(*b*) The inverse of $f(x) = -x$ is the same function.
(*c*) The inverse of $f(x) = \sqrt{x}$ is the function $g(y) = y^2$ (defined for $y \geq 0$).
(*d*) The inverse of $f(x) = 2x - 1$ is the function $g(y) = \dfrac{y+1}{2}$.

Not every function has an inverse function. For example, the function $f(x) = x^2$ does not possess an inverse. Since $f(1) = f(-1) = 1$, an inverse function g would have to satisfy $g(1) = 1$ and $g(1) = -1$, which is impossible. However, if we restrict the function $f(x) = x^2$ to the domain $x \geq 0$, then the function $g(y) = \sqrt{y}$ would be an inverse of f. The condition that a function f must satisfy to have an inverse is that f is *one-to-one*; that is, for any x_1 and x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Notation: The inverse of f is denoted f^{-1}. If $y = f(x)$, we often write $x = f^{-1}(y)$. If f is differentiable, we write, as usual, dy/dx for the derivative $f'(x)$, and dx/dy for the derivative $(f^{-1})'(y)$.

If a function f has an inverse and we are given a formula for $f(x)$, then to find a formula for the inverse f^{-1}, we solve the equation $y = f(x)$ for x in terms of y. For example, given $f(x) = 5x + 2$, set $y = 5x + 2$. Then, $x = \frac{y-2}{5}$, and a formula for the inverse function is $f^{-1}(y) = \frac{y-2}{5}$.

DIFFERENTIATION FORMULA for finding dy/dx given dx/dy:

12. $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$

EXAMPLE 2: Find dy/dx, given $x = \sqrt{y} + 5$.
First method: Solve for $y = (x-5)^2$. Then $dy/dx = 2(x-5)$.

Second method: Differentiate to find $\dfrac{dx}{dy} = \dfrac{1}{2}y^{-1/2} = \dfrac{1}{2\sqrt{y}}$. Then, by rule 12, $\dfrac{dy}{dx} = 2\sqrt{y} = 2(x-5)$.

(See Problems 14, 15, and 57 to 62.)

COMPOSITE FUNCTIONS; THE CHAIN RULE. For two functions f and g, the function given by the formula $f(g(x))$ is called a *composite* function. If f and g are differentiable, then so is the composite function, and its derivative may be obtained by either of two procedures. The first is to compute an explicit formula for $f(g(x))$ and differentiate.

EXAMPLE 3: If $f(x) = x^2 + 3$ and $g(x) = 2x + 1$, then

$$y = f(g(x)) = (2x+1)^2 + 3 = 4x^2 + 4x + 4 \quad \text{and} \quad \frac{dy}{dx} = 8x + 4$$

The derivative of a composite function may also be obtained with the following rule:

13. *The chain rule*: $D_x(f(g(x))) = f'(g(x))g'(x)$

If f is called the *outer function* and g is called the *inner function*, then $D_x(f(g(x)))$ is the product of the derivative of the outer function (evaluated at $g(x)$) and the derivative of the inner function.

EXAMPLE 4: In Example 3, $f'(x) = 2x$ and $g'(x) = 2$. Hence, by the chain rule,

$$D_x(f(g(x))) = f'(g(x))g'(x) = 2g(x) \cdot 2 = 4g(x) = 4(2x+1) = 8x + 4$$

ALTERNATIVE FORMULATION OF THE CHAIN RULE. Write $y = f(u)$ and $u = g(x)$. Then the composite function is $y = f(u) = f(g(x))$, and we have:

The chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

EXAMPLE 5: Let $y = u^3$ and $u = 4x^2 - 2x + 5$. Then the composite function $y = (4x^2 - 2x + 5)^3$ has the derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(8x-2) = 3(4x^2 - 2x + 5)^2(8x - 2)$$

Notes: (1) In the second formulation of the chain rule, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, the y on the left denotes the composite function of x, whereas the y on the right denotes the original function of u (what we called the *outer function* before). (2) Differentiation rule 11 is a special case of the chain rule. (See Problems 16 to 20.)

HIGHER DERIVATIVES. Let $y = f(x)$ be a differentiable function of x, and let its derivative be called the *first derivative* of the function. If the first derivative is differentiable, its derivative is called the *second derivative* of the (original) function and is denoted by one of the symbols $\frac{d^2y}{dx^2}$, y'', or $f''(x)$. In turn, the derivative of the second derivative is called the *third derivative* of the function and is denoted by one of the symbols $\frac{d^3y}{dx^3}$, y''', or $f'''(x)$. And so on.

Note: The derivative of a given order at a point can exist only when the function and all derivatives of lower order are differentiable at the point. (See Problems 21 to 23.)

Solved Problems

1. Prove: (*a*) $\frac{d}{dx}(c) = 0$, where c is any constant; (*b*) $\frac{d}{dx}(x) = 1$; (*c*) $\frac{d}{dx}(cx) = c$, where c is any constant; and (*d*) $\frac{d}{dx}(x^n) = nx^{n-1}$, when n is a positive integer.

Since $\frac{d}{dx} f(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$,

(*a*) $\frac{d}{dx}(c) = \lim\limits_{\Delta x\to 0} \frac{c-c}{\Delta x} = \lim\limits_{\Delta x\to 0} 0 = 0$

(*b*) $\frac{d}{dx}(x) = \lim\limits_{\Delta x\to 0} \frac{(x+\Delta x) - x}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0} 1 = 1$

(*c*) $\frac{d}{dx}(cx) = \lim\limits_{\Delta x\to 0} \frac{c(x+\Delta x) - cx}{\Delta x} = \lim\limits_{\Delta x\to 0} c = c$

(*d*) $$\frac{d}{dx}(x^n) = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1\cdot 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right] - x^n}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\left[nx^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}\right] = nx^{n-1}$$

2. Let u and v be differentiable functions of x. Prove: (*a*) $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$;

(*b*) $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$; (*c*) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$, $v \neq 0$

(*a*) Set $f(x) = u + v = u(x) + v(x)$; then

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} = \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x}$$

Taking the limit as $\Delta x \to 0$ yields $\frac{d}{dx} f(x) = \frac{d}{dx}(u+v) = \frac{d}{dx}u(x) + \frac{d}{dx}v(x) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$.

(*b*) Set $f(x) = uv = u(x)v(x)$; then

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$

$$= \frac{[u(x+\Delta x)v(x+\Delta x) - v(x)u(x+\Delta x)] + [v(x)u(x+\Delta x) - u(x)v(x)]}{\Delta x}$$

$$= u(x+\Delta x)\,\frac{v(x+\Delta x) - v(x)}{\Delta x} + v(x)\,\frac{u(x+\Delta x) - u(x)}{\Delta x}$$

and for $\Delta x \to 0$, $\dfrac{d}{dx} f(x) = \dfrac{d}{dx}(uv) = u(x)\dfrac{d}{dx} v(x) + v(x)\dfrac{d}{dx} u(x) = u\dfrac{d}{dx}(v) + v\dfrac{d}{dx}(u)$.

(*c*) Set $f(x) = \dfrac{u}{v} = \dfrac{u(x)}{v(x)}$; then

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\dfrac{u(x+\Delta x)}{v(x+\Delta x)} - \dfrac{u(x)}{v(x)}}{\Delta x} = \frac{u(x+\Delta x)v(x) - u(x)v(x+\Delta x)}{\Delta x\{v(x)v(x+\Delta x)\}}$$

$$= \frac{[u(x+\Delta x)v(x) - u(x)v(x)] - [u(x)v(x+\Delta x) - u(x)v(x)]}{\Delta x[v(x)v(x+\Delta x)]}$$

$$= \frac{v(x)\,\dfrac{u(x+\Delta x) - u(x)}{\Delta x} - u(x)\,\dfrac{v(x+\Delta x) - v(x)}{\Delta x}}{v(x)v(x+\Delta x)}$$

and for $\Delta x \to 0$, $\dfrac{d}{dx} f(x) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(x)\dfrac{d}{dx} u(x) - u(x)\dfrac{d}{dx} v(x)}{[v(x)]^2} = \dfrac{v\dfrac{d}{dx}(u) - u\dfrac{d}{dx}(v)}{v^2}$

3. Differentiate $y = 4 + 2x - 3x^2 - 5x^3 - 8x^4 + 9x^5$.

$$\frac{dy}{dx} = 0 + 2(1) - 3(2x) - 5(3x^2) - 8(4x^3) + 9(5x^4) = 2 - 6x - 15x^2 - 32x^3 + 45x^4$$

4. Differentiate $y = \dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{2}{x^3} = x^{-1} + 3x^{-2} + 2x^{-3}$.

$$\frac{dy}{dx} = -x^{-2} + 3(-2x^{-3}) + 2(-3x^{-4}) = -x^{-2} - 6x^{-3} - 6x^{-4} = -\frac{1}{x^2} - \frac{6}{x^3} - \frac{6}{x^4}$$

5. Differentiate $y = 2x^{1/2} + 6x^{1/3} - 2x^{3/2}$.

$$\frac{dy}{dx} = 2\left(\frac{1}{2}x^{-1/2}\right) + 6\left(\frac{1}{3}x^{-2/3}\right) - 2\left(\frac{3}{2}x^{1/2}\right) = x^{-1/2} + 2x^{-2/3} - 3x^{1/2} = \frac{1}{x^{1/2}} + \frac{2}{x^{2/3}} - 3x^{1/2}$$

6. Differentiate $y = \dfrac{2}{x^{1/2}} + \dfrac{6}{x^{1/3}} - \dfrac{2}{x^{3/2}} - \dfrac{4}{x^{3/4}} = 2x^{-1/2} + 6x^{-1/3} - 2x^{-3/2} - 4x^{-3/4}$

$$\frac{dy}{dx} = 2\left(-\frac{1}{2}x^{-3/2}\right) + 6\left(-\frac{1}{3}x^{-4/3}\right) - 2\left(-\frac{3}{2}x^{-5/2}\right) - 4\left(-\frac{3}{4}x^{-7/4}\right)$$

$$= -x^{-3/2} - 2x^{-4/3} + 3x^{-5/2} + 3x^{-7/4} = -\frac{1}{x^{3/2}} - \frac{2}{x^{4/3}} + \frac{3}{x^{5/2}} + \frac{3}{x^{7/4}}$$

7. Differentiate $y = \sqrt[3]{3x^2} - \dfrac{1}{\sqrt{5x}} = (3x^2)^{1/3} - (5x)^{-1/2}$.

$$\frac{dy}{dx} = \frac{1}{3}(3x^2)^{-2/3}(6x) - \left(-\frac{1}{2}\right)(5x)^{-3/2}(5) = \frac{2x}{(9x^4)^{1/3}} + \frac{5}{2(5x)(5x)^{1/2}} = \frac{2}{\sqrt[3]{9x}} + \frac{1}{2x\sqrt{5x}}$$

8. Differentiate $s = (t^2 - 3)^4$.

$$\frac{ds}{dt} = 4(t^2-3)^3(2t) = 8t(t^2-3)^3$$

9. Differentiate $z = \dfrac{3}{(a^2-y^2)^2} = 3(a^2-y^2)^{-2}$.

$$\frac{dz}{dy} = 3(-2)(a^2-y^2)^{-3}\frac{d}{dy}(a^2-y^2) = 3(-2)(a^2-y^2)^{-3}(-2y) = \frac{12y}{(a^2-y^2)^3}$$

10. Differentiate $f(x) = \sqrt{x^2+6x+3} = (x^2+6x+3)^{1/2}$.

$$f'(x) = \tfrac{1}{2}(x^2+6x+3)^{-1/2}\frac{d}{dx}(x^2+6x+3) = \tfrac{1}{2}(x^2+6x+3)^{-1/2}(2x+6) = \frac{x+3}{\sqrt{x^2+6x+3}}$$

11. Differentiate $y = (x^2+4)^2(2x^3-1)^3$.

$$\begin{aligned}
y' &= (x^2+4)^2\frac{d}{dx}(2x^3-1)^3 + (2x^3-1)^3\frac{d}{dx}(x^2+4)^2\\
&= (x^2+4)^2(3)(2x^3-1)^2\frac{d}{dx}(2x^3-1) + (2x^3-1)^3(2)(x^2+4)\frac{d}{dx}(x^2+4)\\
&= (x^2+4)^2(3)(2x^3-1)^2(6x^2) + (2x^3-1)^3(2)(x^2+4)(2x)\\
&= 2x(x^2+4)(2x^3-1)^2(13x^3+36x-2)
\end{aligned}$$

12. Differentiate $y = \dfrac{3-2x}{3+2x}$.

$$y' = \frac{(3+2x)\dfrac{d}{dx}(3-2x) - (3-2x)\dfrac{d}{dx}(3+2x)}{(3+2x)^2} = \frac{(3+2x)(-2)-(3-2x)(2)}{(3+2x)^2} = \frac{-12}{(3+2x)^2}$$

13. Differentiate $y = \dfrac{x^2}{\sqrt{4-x^2}} = \dfrac{x^2}{(4-x^2)^{1/2}}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(4-x^2)^{1/2}\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(4-x^2)^{1/2}}{4-x^2} = \frac{(4-x^2)^{1/2}(2x) - (x^2)(\tfrac{1}{2})(4-x^2)^{-1/2}(-2x)}{4-x^2}\\
&= \frac{(4-x^2)^{1/2}(2x) + x^3(4-x^2)^{-1/2}}{4-x^2}\,\frac{(4-x^2)^{1/2}}{(4-x^2)^{1/2}} = \frac{2x(4-x^2)+x^3}{(4-x^2)^{3/2}} = \frac{8x-x^3}{(4-x^2)^{3/2}}
\end{aligned}$$

14. Find dy/dx, given $x = y\sqrt{1-y^2}$.

$$\frac{dx}{dy} = (1-y^2)^{1/2} + \tfrac{1}{2}y(1-y^2)^{-1/2}(-2y) = \frac{1-2y^2}{\sqrt{1-y^2}} \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{\sqrt{1-y^2}}{1-2y^2}$$

15. Find the slope of the curve $x = y^2 - 4y$ at the points where it crosses the y axis.

The points of crossing are $(0,0)$ and $(0,4)$. We have $\dfrac{dx}{dy} = 2y-4$ and so $\dfrac{dy}{dx} = \dfrac{1}{dx/dy} = \dfrac{1}{2y-4}$. At $(0,0)$ the slope is $-\frac{1}{4}$, and at $(0,4)$ the slope is $\frac{1}{4}$.

THE CHAIN RULE

16. Derive the alternative chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Let Δu and Δy be, respectively, the increments given to y and u when x is given an increment Δx. Now, provided $\Delta u \neq 0$, $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u}\,\dfrac{\Delta u}{\Delta x}$; and, provided $\Delta u \neq 0$ as $\Delta x \to 0$, $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$ as required.

The restriction on Δu can usually be met by taking $|\Delta x|$ sufficiently small. When this is not possible, the chain rule may be established as follows:

Set $\Delta y = \dfrac{dy}{du}\,\Delta u + \epsilon\,\Delta u$, where $\epsilon \to 0$ as $\Delta x \to 0$. (See Problem 13 of Chapter 9.) Then

$$\frac{\Delta y}{\Delta x} = \frac{dy}{du}\,\frac{\Delta u}{\Delta x} + \epsilon\,\frac{\Delta u}{\Delta x}$$

and, taking the limits as $\Delta x \to 0$ yields $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx} + 0\,\dfrac{du}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$ as before.

17. Find dy/dx, given $y = \dfrac{u^2 - 1}{u^2 + 1}$ and $u = \sqrt[3]{x^2 + 2}$.

$$\frac{dy}{du} = \frac{4u}{(u^2+1)^2} \qquad \text{and} \qquad \frac{du}{dx} = \frac{2x}{3(x^2+2)^{2/3}} = \frac{2x}{3u^2}$$

Then
$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{4u}{(u^2+1)^2}\,\frac{2x}{3u^2} = \frac{8x}{3u(u^2+1)^2}$$

18. A point moves along the curve $y = x^3 - 3x + 5$ so that $x = \frac{1}{2}\sqrt{t} + 3$, where t is time. At what rate is y changing when $t = 4$?

We are to find the value of dy/dt when $t = 4$. We have

$$\frac{dy}{dx} = 3(x^2 - 1) \qquad \text{and} \qquad \frac{dx}{dt} = \frac{1}{4\sqrt{t}} \qquad \text{so} \qquad \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = \frac{3(x^2-1)}{4\sqrt{t}}$$

When $t = 4$, $x = \frac{1}{2}\sqrt{4} + 3 = 4$, and $\dfrac{dy}{dt} = \dfrac{3(16-1)}{4(2)} = \dfrac{45}{8}$ units per unit of time.

19. A point moves in the plane according to the equations $x = t^2 + 2t$ and $y = 2t^3 - 6t$. Find dy/dx when $t = 0$, 2, and 5.

Since the first relation may be solved for t and this result substituted for t in the second relation, y is clearly a function of x. We have $\dfrac{dy}{dt} = 6t^2 - 6$ and $\dfrac{dx}{dt} = 2t + 2$, from which $\dfrac{dt}{dx} = \dfrac{1}{2t+2}$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = 6(t^2 - 1)\,\frac{1}{2(t+1)} = 3(t-1)$$

The required values of dy/dx are -3 at $t = 0$, 3 at $t = 2$, and 12 at $t = 5$.

20. If $y = x^2 - 4x$ and $x = \sqrt{2t^2 + 1}$, find dy/dt when $t = \sqrt{2}$.

$$\frac{dy}{dx} = 2(x-2) \qquad \text{and} \qquad \frac{dx}{dt} = \frac{2t}{(2t^2+1)^{1/2}} \qquad \text{so} \qquad \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = \frac{4t(x-2)}{(2t^2+1)^{1/2}}$$

When $t = \sqrt{2}$, $x = \sqrt{5}$ and $\dfrac{dy}{dt} = \dfrac{4\sqrt{2}(\sqrt{5}-2)}{\sqrt{5}} = \dfrac{4\sqrt{2}}{5}(5 - 2\sqrt{5})$.

21. Show that the function $f(x) = x^3 + 3x^2 - 8x + 2$ has derivatives of all orders at $x = a$.

$$f'(x) = 3x^2 + 6x - 8 \quad \text{and} \quad f'(a) = 3a^2 + 6a - 8$$
$$f''(x) = 6x + 6 \quad \text{and} \quad f''(a) = 6a + 6$$
$$f'''(x) = 6 \quad \text{and} \quad f'''(a) = 6$$

All derivatives of higher order exist and are identically zero.

22. Investigate the successive derivatives of $f(x) = x^{4/3}$ at $x = 0$.

$$f'(x) = \frac{4}{3}x^{1/3} \quad \text{and} \quad f'(0) = 0$$
$$f''(x) = \frac{4}{9x^{2/3}} \quad \text{and} \quad f''(0) \text{ does not exist}$$

Thus the first derivative, but no derivative of higher order, exists at $x = 0$.

23. Given $f(x) = \dfrac{2}{1-x} = 2(1-x)^{-1}$, find $f^{(n)}(x)$.

$$f'(x) = 2(-1)(1-x)^{-2}(-1) = 2(1-x)^{-2} = 2(1!)(1-x)^{-2}$$
$$f''(x) = 2(1!)(-2)(1-x)^{-3}(-1) = 2(2!)(1-x)^{-3}$$
$$f'''(x) = 2(2!)(-3)(1-x)^{-4}(-1) = 2(3!)(1-x)^{-4}$$

which suggest $f^{(n)}(x) = 2(n!)(1-x)^{-(n+1)}$. This result may be established by mathematical induction by showing that if $f^{(k)}(x) = 2(k!)(1-x)^{-(k+1)}$, then

$$f^{(k+1)}(x) = -2(k!)(k+1)(1-x)^{-(k+2)}(-1) = 2[(k+1)!](1-x)^{-(k+2)}$$

Supplementary Problems

24. Establish formula 10 for $m = -1/n$, n a positive integer, by using formula 9 to compute $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right)$. (For the case $m = p/q$, p and q integers, see Problem 4 of Chapter 11.)

In Problems 25 to 43, find the derivative.

25. $y = x^5 + 5x^4 - 10x^2 + 6$ *Ans.* $dy/dx = 5x(x^3 + 4x^2 - 4)$

26. $y = 3x^{1/2} - x^{3/2} + 2x^{-1/2}$ *Ans.* $dy/dx = \dfrac{3}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} - 1/x^{3/2}$

27. $y = \dfrac{1}{2x^2} + \dfrac{4}{\sqrt{x}} = \dfrac{1}{2}x^{-2} + 4x^{-1/2}$ *Ans.* $\dfrac{dy}{dx} = -\dfrac{1}{x^3} - \dfrac{2}{x^{3/2}}$

28. $y = \sqrt{2x} + 2\sqrt{x}$ *Ans.* $y' = (1 + \sqrt{2})/\sqrt{2x}$

29. $f(t) = \dfrac{2}{\sqrt{t}} + \dfrac{6}{\sqrt[3]{t}}$ *Ans.* $f'(t) = -\dfrac{t^{1/2} + 2t^{2/3}}{t^2}$

30. $y = (1 - 5x)^6$ *Ans.* $y' = -30(1 - 5x)^5$

31. $f(x) = (3x - x^3 + 1)^4$ *Ans.* $f'(x) = 12(1 - x^2)(3x - x^3 + 1)^3$

32. $y = (3 + 4x - x^2)^{1/2}$ *Ans.* $y' = (2 - x)/y$

33. $\theta = \dfrac{3r + 2}{2r + 3}$ *Ans.* $\dfrac{d\theta}{dr} = \dfrac{5}{(2r + 3)^2}$

34. $y = \left(\dfrac{x}{1+x}\right)^5$ *Ans.* $y' = \dfrac{5x^4}{(1+x)^6}$

35. $y = 2x^2\sqrt{2-x}$ *Ans.* $y' = \dfrac{x(8-5x)}{\sqrt{2-x}}$

36. $f(x) = x\sqrt{3-2x^2}$ *Ans.* $f'(x) = \dfrac{3-4x^2}{\sqrt{3-2x^2}}$

37. $y = (x-1)\sqrt{x^2-2x+2}$ *Ans.* $\dfrac{dy}{dx} = \dfrac{2x^2-4x+3}{\sqrt{x^2-2x+2}}$

38. $z = \dfrac{w}{\sqrt{1-4w^2}}$ *Ans.* $\dfrac{dz}{dw} = \dfrac{1}{(1-4w^2)^{3/2}}$

39. $y = \sqrt{1+\sqrt{x}}$ *Ans.* $y' = \dfrac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$

40. $f(x) = \sqrt{\dfrac{x-1}{x+1}}$ *Ans.* $f'(x) = \dfrac{1}{(x+1)\sqrt{x^2-1}}$

41. $y = (x^2+3)^4(2x^3-5)^3$ *Ans.* $y' = 2x(x^2+3)^3(2x^3-5)^2(17x^3+27x-20)$

42. $s = \dfrac{t^2+2}{3-t^2}$ *Ans.* $\dfrac{ds}{dt} = \dfrac{10t}{(3-t^2)^2}$

43. $y = \left(\dfrac{x^3-1}{2x^3+1}\right)^4$ *Ans.* $y' = \dfrac{36x^2(x^3-1)^3}{(2x^3+1)^5}$

44. For each of the following, compute dy/dx by two different methods and check that the results are the same: (*a*) $x = (1+2y)^3$, (*b*) $x = 1/(2+y)$.

In Problems 45 to 48, use the chain drule to find dy/dx.

45. $y = \dfrac{u-1}{u+1}$, $u = \sqrt{x}$ *Ans.* $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}$

46. $y = u^3 + 4$, $u = x^2 + 2x$ *Ans.* $dy/dx = 6x^2(x+2)^2(x+1)$

47. $y = \sqrt{1+u}$, $u = \sqrt{x}$ *Ans.* See Problem 39.

48. $y = \sqrt{u}$, $u = v(3-2v)$, $v = x^2$ $\left(\textit{Hint: } \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dx}.\right)$ *Ans.* See Problem 36.

In Problems 49 to 52, find the indicated derivative.

49. $y = 3x^4 - 2x^2 + x - 5$; y''' *Ans.* $y''' = 72x$

50. $y = 1/\sqrt{x}$; $y^{(\text{iv})}$ *Ans.* $y^{(\text{iv})} = \dfrac{105}{16x^{9/2}}$

51. $f(x) = \sqrt{2-3x^2}$; $f''(x)$ *Ans.* $f''(x) = -6/(2-3x^2)^{3/2}$

52. $y = x/\sqrt{x-1}$, y'' *Ans.* $y'' = \dfrac{4-x}{4(x-1)^{5/2}}$

In Problems 53 and 54, find the nth derivative.

53. $y = 1/x^2$ *Ans.* $y^{(n)} = \dfrac{(-1)^n[(n+1)!]}{x^{n+2}}$

54. $f(x) = 1/(3x+2)$ *Ans.* $f^{(n)}(x) = (-1)^n \dfrac{3^n(n!)}{(3x+2)^{n+1}}$

55. If $y = f(u)$ and $u = g(x)$, show that

(a) $\dfrac{d^2y}{dx^2} = \dfrac{dy}{du} \cdot \dfrac{d^2u}{dx^2} + \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2$ (b) $\dfrac{d^3y}{dx^3} = \dfrac{dy}{du} \cdot \dfrac{d^3u}{dx^3} + 3\dfrac{d^2y}{du^2} \cdot \dfrac{d^2u}{dx^2} \cdot \dfrac{du}{dx} + \dfrac{d^3y}{du^3}\left(\dfrac{du}{dx}\right)^3$

56. From $\dfrac{dx}{dy} = \dfrac{1}{y'}$, derive $\dfrac{d^2x}{dy^2} = -\dfrac{y''}{(y')^3}$ and $\dfrac{d^3x}{dy^3} = \dfrac{3(y'')^2 - y'y'''}{(y')^5}$.

In Problems 57 to 62, determine whether the given function f has an inverse; if it does, find a formula for the inverse f^{-1} and calculate its derivative.

57. $f(x) = 1/x$ *Ans.* $x = f^{-1}(y) = 1/y$; $dx/dy = -x^2 = -1/y^2$

58. $f(x) = \frac{1}{3}x + 4$ *Ans.* $x = f^{-1}(y) = 3y - 12$; $dx/dy = 3$

59. $f(x) = \sqrt{x-5}$ *Ans.* $x = f^{-1}(y) = y^2 + 5$; $dx/dy = 2y = 2\sqrt{x-5}$

60. $f(x) = x^2 + 2$ *Ans.* no inverse function

61. $f(x) = x^3$ *Ans.* $x = f^{-1}(y) = \sqrt[3]{y}$; $\dfrac{dx}{dy} = \dfrac{1}{3x^2} = \dfrac{1}{3}y^{-2/3}$

62. $f(x) = \dfrac{2x-1}{x+2}$ *Ans.* $x = f^{-1}(y) = -\dfrac{2y+1}{y-2}$; $\dfrac{dx}{dy} = \dfrac{5}{(y-2)^2}$

Chapter 11

Implicit Differentiation

IMPLICIT FUNCTIONS. An equation $f(x, y) = 0$, on perhaps certain restricted ranges of the variables, is said to define y *implicitly* as a function of x.

EXAMPLE 1: (*a*) The equation $xy + x - 2y - 1 = 0$, with $x \neq 2$, defines the function $y = \frac{1-x}{x-2}$.
(*b*) The equation $4x^2 + 9y^2 - 36 = 0$ defines the function $y = \frac{2}{3}\sqrt{9 - x^2}$ when $|x| \leq 3$ and $y \geq 0$, and the function $y = -\frac{2}{3}\sqrt{9 - x^2}$ when $|x| \leq 3$ and $y \leq 0$. The ellipse determined by the given equation should be thought of as consisting of two arcs joined at the points $(-3, 0)$ and $(3, 0)$.

The derivative y' may be obtained by one of the following procedures:

1. Solve, when possible, for y and differentiate with respect to x. Except for very simple equations, this procedure is to be avoided.
2. Thinking of y as a function of x, differentiate both sides of the given equation with respect to x and solve the resulting relation for y'. This differentiation process is known as *implicit differentiation*.

EXAMPLE 2: (*a*) Find y', given $xy + x - 2y - 1 = 0$.
We have $x\frac{d}{dx}(y) + y\frac{d}{dx}(x) + \frac{d}{dx}(x) - 2\frac{d}{dx}(y) - \frac{d}{dx}(1) = \frac{d}{dx}(0)$
or $xy' + y + 1 - 2y' = 0$; then $y' = \frac{1+y}{2-x}$.
(*b*) Find y' when $x = \sqrt{5}$, given $4x^2 + 9y^2 - 36 = 0$.
We have $4\frac{d}{dx}(x^2) + 9\frac{d}{dx}(y^2) = 8x + 9\frac{d}{dy}(y^2)\frac{dy}{dx} = 8x + 18yy' = 0$
or $y' = -4x/9y$. When $x = \sqrt{5}$, $y = \pm 4/3$. At the point $(\sqrt{5}, 4/3)$ on the upper arc of the ellipse, $y' = -\sqrt{5}/3$, and at the point $(\sqrt{5}, -4/3)$ on the lower arc, $y' = \sqrt{5}/3$.

DERIVATIVES OF HIGHER ORDER may be obtained in two ways. The first is to differentiate implicitly the derivative of one lower order and replace y' by the relation previously found.

EXAMPLE 3: From Example 2(*a*), $y' = \frac{1+y}{2-x}$. Then

$$\frac{d}{dx}(y') = y'' = \frac{d}{dx}\left(\frac{1+y}{2-x}\right) = \frac{(2-x)y' + 1 + y}{(2-x)^2} = \frac{(2-x)\left(\frac{1+y}{2-x}\right) + 1 + y}{(2-x)^2} = \frac{2+2y}{(2-x)^2}.$$

The second method is to differentiate implicitly both sides of the given equation as many times as is necessary to produce the required derivative and eliminate all derivatives of lower order. This procedure is recommended only when a derivative of higher order at a given point is required.

EXAMPLE 4: Find the value of y'' at the point $(-1, 1)$ of the curve $x^2y + 3y - 4 = 0$.
We differentiate implicitly with respect to x twice, obtaining

$$x^2y' + 2xy + 3y' = 0 \qquad \text{and} \qquad x^2y'' + 2xy' + 2xy' + 2y + 3y'' = 0$$

We substitute $x = -1$, $y = 1$ in the first relation to obtain $y' = \frac{1}{2}$. Then we substitute $x = -1$, $y = 1$, $y' = \frac{1}{2}$ in the second relation to get $y'' = 0$.

Solved Problems

1. Find y', given $x^2y - xy^2 + x^2 + y^2 = 0$.

$$\frac{d}{dx}(x^2y) - \frac{d}{dx}(xy^2) + \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) - x\frac{d}{dx}(y^2) - y^2\frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Hence $\quad x^2y' + 2xy - 2xyy' - y^2 + 2x + 2yy' = 0 \quad$ and $\quad y' = \dfrac{y^2 - 2x - 2xy}{x^2 + 2y - 2xy}$

2. Find y' and y'', given $x^2 - xy + y^2 = 3$.

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 2x - xy' - y + 2yy' = 0. \qquad \text{So} \qquad y' = \frac{2x - y}{x - 2y}$$

Then $$y'' = \frac{(x-2y)\dfrac{d}{dx}(2x-y) - (2x-y)\dfrac{d}{dx}(x-2y)}{(x-2y)^2} = \frac{(x-2y)(2-y') - (2x-y)(1-2y')}{(x-2y)^2}$$

$$= \frac{3xy' - 3y}{(x-2y)^2} = \frac{3x\left(\dfrac{2x-y}{x-2y}\right) - 3y}{(x-2y)^2} = \frac{6(x^2 - xy + y^2)}{(x-2y)^3} = \frac{18}{(x-2y)^3}$$

3. Find y' and y'', given $x^3y + xy^3 = 2$ and $x = 1$.

We have

$$x^3y' + 3x^2y + 3xy^2y' + y^3 = 0$$

and $$x^3y'' + 3x^2y' + 3x^2y' + 6xy + 3xy^2y'' + 6xy(y')^2 + 3y^2y' + 3y^2y' = 0$$

When $x = 1$, $y = 1$; substituting these values in the first derived relation yields $y' = -1$. Then substituting $x = 1$, $y = 1$, $y' = -1$ in the second relation yields $y'' = 0$.

Supplementary Problems

4. Establish formula 10 of Chapter 10 for $m = p/q$, p and q integers, by writing $y = x^{p/q}$ as $y^q = x^p$ and differentiating with respect to x.

5. Find y'', given (*a*) $x + xy + y = 2$; (*b*) $x^3 - 3xy + y^3 = 1$.

Ans. (*a*) $y'' = \dfrac{2(1+y)}{(1+x)^2}$; (*b*) $y'' = -\dfrac{4xy}{(y^2-x)^3}$

6. Find y', y'', and y''' at (*a*) the point $(2, 1)$ on $x^2 - y^2 - x = 1$; (*b*) the point $(1, 1)$ on $x^3 + 3x^2y - 6xy^2 + 2y^3 = 0$. *Ans.* (*a*) $3/2$, $-5/4$, $45/8$; (*b*) 1, 0, 0

7. Find the slope at the point (x_0, y_0) of (*a*) $b^2x^2 + a^2y^2 = a^2b^2$; (*b*) $b^2x^2 - a^2y^2 = a^2b^2$; (*c*) $x^3 + y^3 - 6x^2y = 0$.

Ans. (*a*) $-\dfrac{b^2x_0}{a^2y_0}$; (*b*) $\dfrac{b^2x_0}{a^2y_0}$; (*c*) $\dfrac{4x_0y_0 - x_0^2}{y_0^2 - 2x_0^2}$

8. Prove that the lines tangent to the curves $5y - 2x + y^3 - x^2y = 0$ and $2y + 5x + x^4 - x^3y^2 = 0$ at the origin intersect at right angles.

9. (*a*) The total surface area of a rectangular parallelepiped of square base y on a side and height x is given by $S = 2y^2 + 4xy$. If S is constant, find dy/dx without solving for y.

(*b*) The total surface area of a right circular cylinder of radius r and height h is given by $S = 2\pi r^2 + 2\pi rh$. If S is constant, find dr/dh.

Ans. (*a*) $-\dfrac{y}{x+y}$; (*b*) $-\dfrac{r}{2r+h}$

10. For the circle $x^2 + y^2 = r^2$, show that $\left|\dfrac{y''}{[1+(y')^2]^{3/2}}\right| = \dfrac{1}{r}$.

11. Given $S = \pi x(x + 2y)$ and $V = \pi x^2 y$, show that $dS/dx = 2\pi(x - y)$ when V is a constant and $dV/dx = -\pi x(x - y)$ when S is a constant.

Chapter 12

Tangents and Normals

IF THE FUNCTION $f(x)$ has a finite derivative $f'(x_0)$ at $x = x_0$, the curve $y = f(x)$ has a tangent at $P_0(x_0, y_0)$ whose slope is

$$m = \tan\theta = f'(x_0)$$

If $m = 0$, the curve has a horizontal tangent of equation $y = y_0$ at P_0, as at A, C, and E of Fig. 2-1. Otherwise the equation of the tangent is

$$y - y_0 = m(x - x_0)$$

If $f(x)$ is continuous at $x = x_0$ but $\lim_{x \to x_0} f'(x) = \infty$, the curve has a vertical tangent of equation $x = x_0$, as at B and D of Fig. 12-1.

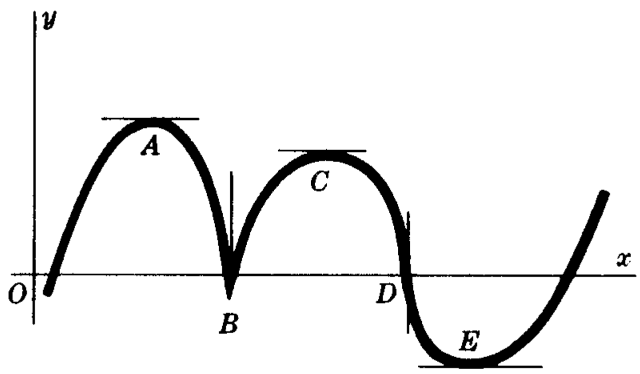


Fig. 12-1

The *normal* to a curve at one of its points is the line that passes through the point and is perpendicular to the tangent at the point. The equation of the normal at $P_0(x_0, y_0)$ is

$$x = x_0 \text{ if the tangent is horizontal}$$
$$y = y_0 \text{ if the tangent is vertical}$$
$$y - y_0 = -\frac{1}{m}(x - x_0) \text{ otherwise}$$

(See Problems 1 to 8.)

THE ANGLE OF INTERSECTION of two curves is defined as the angle between the tangents to the curve at their point of intersection.

To determine the angles of intersection of two curves:

1. Solve the equations simultaneously to find the points of intersection.
2. Find the slopes m_1 and m_2 of the tangents to the two curves at each point of intersection.
3. If $m_1 = m_2$, the angle of intersection is $\phi = 0°$, and if $m_1 = -1/m_2$, the angle of intersection is $\phi = 90°$; otherwise it can be found from

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

ϕ is the *acute* angle of intersection when $\tan\phi > 0$, and $180° - \phi$ is the acute angle of intersection when $\tan\phi < 0$.

(See Problems 9 to 11.)

Solved Problems

1. Find the points of tangency of horizontal and vertical tangents to the curve $x^2 - xy + y^2 = 27$.

Differentiating yields $y' = \dfrac{y-2x}{2y-x}$.

For horizontal tangents: Set the numerator of y' equal to zero and obtain $y = 2x$. The points of tangency are the points of intersection of the line $y = 2x$ and the given curve. Simultaneously solve the two equations to find that these points are $(3, 6)$ and $(-3, -6)$.

For vertical tangents: Set the denominator of y' equal to zero and obtain $x = 2y$. The points of tangency are the points of intersection of the line $x = 2y$ and the given curve. Simultaneously solve the two equations to find that these points are $(6, 3)$ and $(-6, -3)$.

2. Find the equations of the tangent and normal to $y = x^3 - 2x^2 + 4$ at $(2, 4)$.

$f'(x) = 3x^2 - 4x$; hence the slope of the tangent at $(2, 4)$ is $m = f'(2) = 4$.
The equation of the tangent is $y - 4 = 4(x-2)$ or $y = 4x - 4$.
The equation of the normal is $y - 4 = -\frac{1}{4}(x-2)$ or $x + 4y = 18$.

3. Find the equations of the tangent and normal to $x^2 + 3xy + y^2 = 5$ at $(1, 1)$.

$\dfrac{dy}{dx} = -\dfrac{2x+3y}{3x+2y}$; hence the slope of the tangent at $(1, 1)$ is $m = -1$.
The equation of the tangent is $y - 1 = -1(x-1)$ or $x + y = 2$.
The equation of the normal is $y - 1 = 1(x-1)$ or $x - y = 0$.

4. Find the equations of the tangents with slope $m = -\frac{2}{9}$ to the ellipse $4x^2 + 9y^2 = 40$.

Let $P_0(x_0, y_0)$ be the point of tangency of a required tangent. P_0 is on the ellipse, so

$$4x_0^2 + 9y_0^2 = 40 \tag{1}$$

Also, $\dfrac{dy}{dx} = -\dfrac{4x}{9y}$. Hence, at (x_0, y_0), $m = -\dfrac{4x_0}{9y_0} = -\dfrac{2}{9}$. So $y_0 = 2x_0$. The points of tangency are the simultaneous solutions $(1, 2)$ and $(-1, -2)$ of (*1*) and the equation $y_0 = 2x_0$.

The equation of the tangent at $(1, 2)$ is $y - 2 = -\frac{2}{9}(x-1)$ or $2x + 9y = 20$.
The equation of the tangent at $(-1, -2)$ is $y + 2 = -\frac{2}{9}(x+1)$ or $2x + 9y = -20$.

5. Find the equation of the tangent, through the point $(2, -2)$, to the hyperbola $x^2 - y^2 = 16$.

Let $P_0(x_0, y_0)$ be the point of tangency of the required tangent. P_0 is on the hyperbola, so

$$x_0^2 - y_0^2 = 16 \tag{1}$$

Also, $\dfrac{dy}{dx} = \dfrac{x}{y}$. Hence, at (x_0, y_0), $m = \dfrac{x_0}{y_0} = \dfrac{y_0+2}{x_0-2}$ = slope of the line joining P_0 and $(2, -2)$; then

$$2x_0 + 2y_0 = x_0^2 - y_0^2 = 16 \quad \text{or} \quad x_0 + y_0 = 8 \tag{2}$$

The point of tangency is the simultaneous solution $(5, 3)$ of (*1*) and (*2*). Thus the equation of the tangent is $y - 3 = \frac{5}{3}(x-5)$ or $5x - 3y = 16$.

6. Find the equations of the vertical lines that meet the curves (*1*) $y = x^3 + 2x^2 - 4x + 5$ and (*2*) $3y = 2x^3 + 9x^2 - 3x - 3$ in points at which the tangents to the respective curves are parallel.

Let $x = x_0$ be such a vertical line. The tangents to the curves at x_0 have the slopes

For (*1*): $y' = 3x^2 + 4x - 4$; at $x = x_0$, $m_1 = 3x_0^2 + 4x_0 - 4$
For (*2*): $3y' = 6x^2 + 18x - 3$; at $x = x_0$, $m_2 = 2x_0^2 + 6x_0 - 1$

Since $m_1 = m_2$, we have $3x_0^2 + 4x_0 - 4 = 2x_0^2 + 6x_0 - 1$, from which $x_0 = -1$ and $x_0 = 3$. The lines are $x = -1$ and $x = 3$.

7. (*a*) Show that the equation of the tangent of slope $m \neq 0$ to the parabola $y^2 = 4px$ is $y = mx + p/m$.

(*b*) Show that the equation of the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at the point $P_0(x_0, y_0)$ on the ellipse is $b^2x_0x + a^2y_0y = a^2b^2$.

(*a*) $y' = 2p/y$. Let $P_0(x_0, y_0)$ be the point of tangency; then $y_0^2 = 4px_0$ and $m = 2p/y_0$. Hence, $y_0 = 2p/m$ and $x_0 = \frac{1}{4}y_0^2/p = p/m^2$. The equation of the tangent is then $y - 2p/m = m(x - p/m^2)$ or $y = mx + p/m$.

(*b*) $y' = -\dfrac{b^2x}{a^2y}$. At P_0, $m = -\dfrac{b^2x_0}{a^2y_0}$, and the equation of the tangent is $y - y_0 = -\dfrac{b^2x_0}{a^2y_0}(x - x_0)$ or $b^2x_0x + a^2y_0y = b^2x_0^2 + a^2y_0^2 = a^2b^2$.

8. Show that at a point $P_0(x_0, y_0)$ on the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, the tangent bisects the angle included by the focal radii of P_0.

At P_0 the slope of the tangent to the hyperbola is b^2x_0/a^2y_0 and the slopes of the focal radii P_0F' and P_0F (see Fig. 12-2) are $y_0/(x_0 + c)$ and $y_0/(x_0 - c)$, respectively. Now

$$\tan\alpha = \frac{\dfrac{b^2x_0}{a^2y_0} - \dfrac{y_0}{x_0 + c}}{1 + \dfrac{b^2x_0}{a^2y_0}\dfrac{y_0}{x_0 + c}} = \frac{(b^2x_0^2 - a^2y_0^2) + b^2cx_0}{(a^2 + b^2)x_0y_0 + a^2cy_0} = \frac{a^2b^2 + b^2cx_0}{c^2x_0y_0 + a^2cy_0} = \frac{b^2(a^2 + cx_0)}{cy_0(a^2 + cx_0)} = \frac{b^2}{cy_0}$$

since $b^2x_0^2 - a^2y_0^2 = a^2b^2$ and $a^2 + b^2 = c^2$, and

$$\tan\beta = \frac{\dfrac{y_0}{x_0 - c} - \dfrac{b^2x_0}{a^2y_0}}{1 + \dfrac{b^2x_0}{a^2y_0}\cdot\dfrac{y_0}{x_0 - c}} = \frac{b^2cx_0 - (b^2x_0^2 - a^2y_0^2)}{(a^2 + b^2)x_0y_0 - a^2cy_0} = \frac{b^2cx_0 - a^2b^2}{c^2x_0y_0 - a^2cy_0} = \frac{b^2}{cy_0}$$

Hence, $a = \beta$ because $\tan\alpha = \tan\beta$.

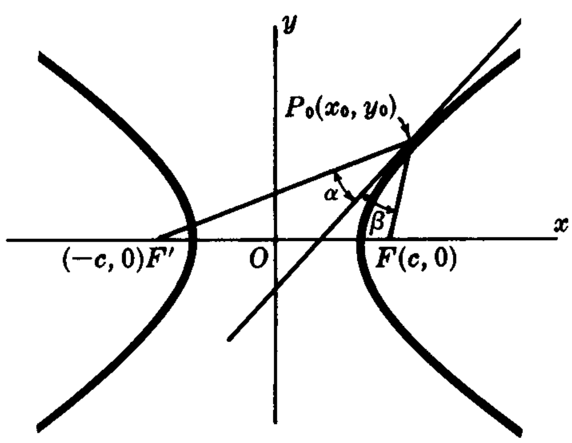


Fig. 12-2

9. Find the acute angles of intersection of the curves (*1*) $y^2 = 4x$ and (*2*) $2x^2 = 12 - 5y$.

The points of intersection of the curves are $P_a(1, 2)$ and $P_b(4, -4)$.
For (*1*), $y' = 2/y$; for (*2*), $y' = -4x/5$. Hence,

At P_a: $m_1 = 1$ and $m_2 = -\frac{4}{5}$, so $\tan\phi = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{1 + 4/5}{1 - 4/5} = 9$ and $\phi = 83°40'$ is the acute angle of intersection.

At P_b: $m_1 = -\frac{1}{2}$ and $m_2 = -\frac{16}{5}$, so $\tan\phi = \dfrac{-1/2 + 16/5}{1 + 8/5} = 1.0385$ and $\phi = 46°5'$ is the acute angle of intersection.

10. Find the acute angles of intersection of the curves (*1*) $2x^2 + y^2 = 20$ and (*2*) $4y^2 - x^2 = 8$.

The points of intersection are $(\pm 2\sqrt{2}, 2)$ and $(\pm 2\sqrt{2}, -2)$.
For (*1*), $y' = -2x/y$; for (*2*), $y' = x/4y$.
At the point $(2\sqrt{2}, 2)$, $m_1 = -2\sqrt{2}$ and $m_2 = \frac{1}{4}\sqrt{2}$. Since $m_1 m_2 = -1$, the angle of intersection is $\phi = 90°$ (i.e., the curves are *orthogonal*). By symmetry, the curves are orthogonal at each of their points of intersection.

11. A cable of a certain suspension bridge is attached to supporting pillars 250 ft apart. If it hangs in the form of a parabola with the lowest point 50 ft below the point of suspension, find the angle between the cable and the pillar.

Take the origin at the vertex of the parabola, as in Fig. 12-3. The equation of the parabola is $y = \frac{2}{625}x^2$, and $y' = 4x/625$.
At $(125, 50)$, $m = 4(125)/625 = 0.8000$ and $\theta = 38°40'$. Hence, the required angle is $\phi = 90° - \theta = 51°20'$.

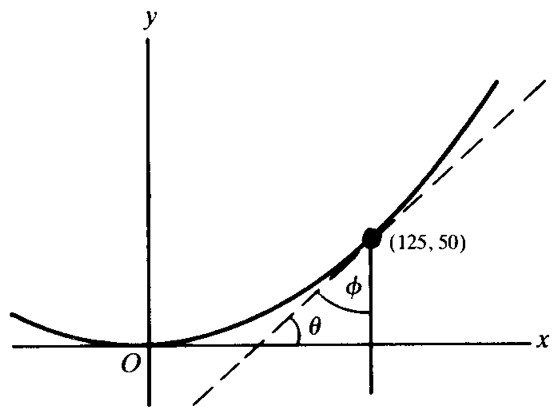


Fig. 12-3

Supplementary Problems

12. Examine $x^2 + 4xy + 16y^2 = 27$ for horizontal and vertical tangents.

Ans. horizontal tangents at $(3, -3/2)$ and $(-3, 3/2)$; vertical tangents at $(6, -3/4)$ and $(-6, 3/4)$

13. Find the equations of the tangent and normal to $x^2 - y^2 = 7$ at the point $(4, -3)$.

Ans. $4x + 3y = 7$; $3x - 4y = 24$

14. At what points on the curve $y = x^3 + 5$ is its tangent (*a*) parallel to the line $12x - y = 17$; (*b*) perpendicular to the line $x + 3y = 2$? *Ans.* (*a*) $(2, 13)$, $(-2, -3)$; (*b*) $(1, 6)$, $(-1, 4)$

15. Find the equations of the tangents to $9x^2 + 16y^2 = 52$ that are parallel to the line $9x - 8y = 1$.

Ans. $9x - 8y = \pm 26$

16. Find the equations of the tangents to the hyperbola $xy = 1$ through the point $(-1, 1)$.

Ans. $y = (2\sqrt{2} - 3)x + 2\sqrt{2} - 2$; $y = -(2\sqrt{2} + 3)x - 2\sqrt{2} - 2$

17. For the parabola $y^2 = 4px$, show that the equation of the tangent at one of its points $P(x_0, y_0)$ is $yy_0 = 2p(x + x_0)$.

18. For the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, show that the equations of its tangents of slope m are $y = mx \pm \sqrt{a^2m^2 + b^2}$.

19. For the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, show that (*a*) the equation of the tangent at one of its points $P(x_0, y_0)$ is $b^2x_0x - a^2y_0y = a^2b^2$ and (*b*) the equations of its tangents of slope m are $y = mx \pm \sqrt{a^2m^2 - b^2}$.

20. Show that the normal to a parabola at any of its points P_0 bisects the angle included by the focal radius of P_0 and the line through P_0 parallel to the axis of the parabola.

21. Prove: Any tangent to a parabola, except at the vertex, intersects the directrix and the latus rectum (produced if necessary) in points equidistant from the focus.

22. Prove: The chord joining the points of contact of the tangents to a parabola through any point on its directrix passes through the focus.

23. Prove: The normal to an ellipse at any of its points P_0 bisects the angle included by the focal radii of P_0.

24. Prove: The point of contact of a tangent of a hyperbola is the midpoint of the segment of the tangent included between the asymptotes.

25. Prove: (*a*) The sum of the intercepts on the coordinate axes of any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is a constant. (*b*) The sum of the squares of the intercepts on the coordinate axes of any tangent to $x^{2/3} + y^{2/3} = a^{2/3}$ is a constant.

26. Find the acute angles of intersection of the circles $x^2 - 4x + y^2 = 0$ and $x^2 + y^2 = 8$. *Ans.* $45°$

27. Show that the curves $y = x^3 + 2$ and $y = 2x^2 + 2$ have a common tangent at the point $(0, 2)$ and intersect at an angle $\phi = \text{Arctan}\,\frac{4}{97}$ at the point $(2, 10)$.

28. Show that the ellipse $4x^2 + 9y^2 = 45$ and the hyperbola $x^2 - 4y^2 = 5$ are orthogonal.

29. Find the equations of the tangent and normal to the parabola $y = 4x^2$ at the point $(-1, 4)$.

Ans. $y + 8x + 4 = 0$; $8y - x - 33 = 0$

30. At what points on the curve $y = 2x^3 + 13x^2 + 5x + 9$ does its tangent pass through the origin?

Ans. $x = -3, -1, 3/4$

Chapter 13

Maximum and Minimum Values

INCREASING AND DECREASING FUNCTIONS. A function $f(x)$ is said to be *increasing* on an open interval if $u < v$ implies $f(u) < f(v)$ for all u and v in the interval. A function $f(x)$ is said to be *increasing at* $x = x_0$ if $f(x)$ is increasing on an open interval containing x_0. Similarly, $f(x)$ is *decreasing* on an open interval if $u < v$ implies $f(u) > f(v)$ for all u and v in the interval, and $f(x)$ is *decreasing at* $x = x_0$ if $f(x)$ is decreasing on an open interval containing x_0.

If $f'(x_0) > 0$, then it can be shown that $f(x)$ is an increasing function at $x = x_0$; similarly, if $f'(x_0) < 0$, then $f(x)$ is a decreasing function at $x = x_0$. (For a proof, see Problem 17.) If $f'(x_0) = 0$, then $f(x)$ is said to be *stationary* at $x = x_0$.

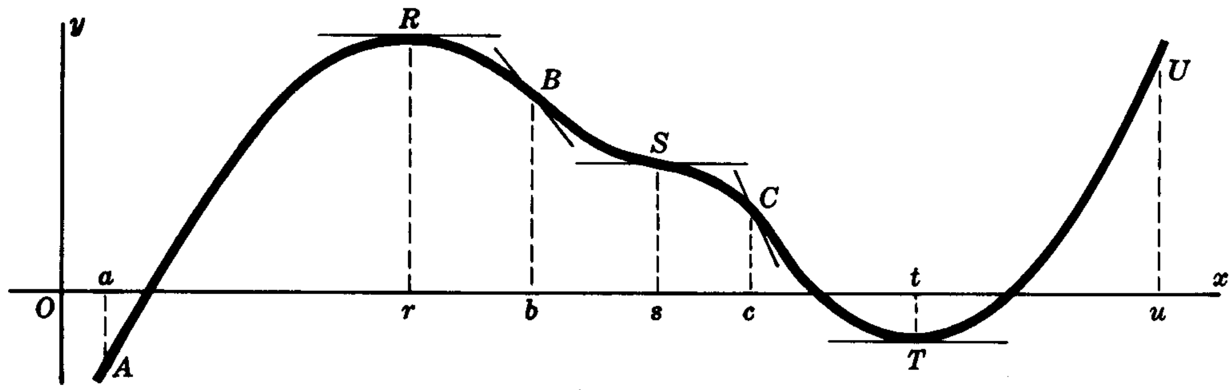


Fig. 13-1

In Fig. 13-1, the curve $y = f(x)$ is rising (the function is increasing) on the intervals $a < x < r$ and $t < x < u$; the curve is falling (the function is decreasing) on the interval $r < x < t$. The function is stationary at $x = r$, $x = s$, and $x = t$; the curve has a horizontal tangent at the points R, S, and T. The values of x (that is, r, s, and t), for which the function $f(x)$ is stationary (that is, for $f'(x) = 0$) are frequently called *critical values* (or *critical numbers*) for the function, and the corresponding points (R, S, and T) of the graph are called *critical points* of the curve.

RELATIVE MAXIMUM AND MINIMUM VALUES OF A FUNCTION. A function $f(x)$ is said to have a *relative maximum* at $x = x_0$ if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0, that is, if the value of $f(x_0)$ is greater than or equal to the values of $f(x)$ at all nearby points. A function $f(x)$ is said to have a *relative minimum* at $x = x_0$ if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0, that is, if the value of $f(x_0)$ is less than or equal to the values of $f(x)$ at all nearby points. (See Problem 1.)

In Fig. 13-1, $R(r, f(r))$ is a relative maximum point of the curve since $f(r) > f(x)$ on any sufficiently small neighborhood $0 < |x - r| < \delta$. We say that $y = f(x)$ has a *relative maximum value* $(=f(r))$ when $x = r$. In the same figure, $T(t, f(t))$ is a relative minimum point of the curve since $f(t) < f(x)$ on any sufficiently small neighborhood $0 < |x - t| < \delta$. We say that $y = f(x)$ has a *relative minimum value* $(=f(t))$ when $x = t$. Note that R joins an arc AR which is rising $(f'(x) > 0)$ and an arc RB which is falling $(f'(x) < 0)$, while T joins an arc CT which is falling $(f'(x) < 0)$ and an arc TU which is rising $(f'(x) > 0)$. At S two arcs BS and SC, both of which are falling, are joined; S is neither a relative maximum point nor a relative minimum point of the curve.

If $f(x)$ is differentiable on $a \leq x \leq b$ and if $f(x)$ has a relative maximum (minimum) value at $x = x_0$, where $a < x_0 < b$, then $f'(x_0) = 0$. For a proof, see Problem 18.

FIRST-DERIVATIVE TEST. The following steps can be used to find the relative maximum (or minimum) values (hereafter called simply maximum [or minimum] values) of a function $f(x)$ that, together with its first derivative, is continuous.

1. Solve $f'(x) = 0$ for the critical values.
2. Locate the critical values on the x axis, thereby establishing a number of intervals.
3. Determine the sign of $f'(x)$ on each interval.
4. Let x increase through each critical value $x = x_0$; then:

   $f(x)$ has a maximum value $f(x_0)$ if $f'(x)$ changes from + to − (Fig. 13-2(*a*)).

   $f(x)$ has a minimum value $f(x_0)$ if $f'(x)$ changes from − to + (Fig. 13-2(*b*)).

   $f(x)$ has neither a maximum nor a minimum value at $x = x_0$ if $f'(x)$ does not change sign (Fig. 13-2(*c*) and (*d*)).

(See Problems 2 to 5.)

A function $f(x)$, necessarily less simple than those of Problems 2 to 5, may have a maximum or minimum value $f(x_0)$ although $f'(x_0)$ does not exist. The values $x = x_0$ for which $f(x)$ is defined but $f'(x)$ does not exist will also be called critical values for the function. They, together with the values for which $f'(x) = 0$, are to be used as the critical values in the first-derivative test. (See Problems 6 to 8.)

CONCAVITY. An arc of a curve $y = f(x)$ is called *concave upward* if, at each of its points, the arc lies above the tangent at that point. As x increases, $f'(x)$ either is of the same sign and increasing (as on the interval $b < x < s$ of Fig. 13-1) or changes sign from negative to positive (as on the interval $c < x < u$). In either case, the slope $f'(x)$ is increasing and $f''(x) > 0$.

An arc of a curve $y = f(x)$ is called *concave downward* if, at each of its points, the arc lies below the tangent at that point. As x increases, $f'(x)$ either is of the same sign and decreasing (as on the interval $s < x < c$) or changes sign from positive to negative (as on the interval $a < x < b$). In either case, the slope $f'(x)$ is decreasing and $f''(x) < 0$.

A POINT OF INFLECTION is a point at which a curve changes from concave upward to concave downward, or vice versa. In Fig. 13-1, the points of inflection are B, S, and C.

A curve $y = f(x)$ has one of its points $x = x_0$ as an inflection point if $f''(x_0) = 0$ or is not defined *and* $f''(x)$ changes sign as x increases through $x = x_0$. The latter condition may be replaced by $f'''(x_0) \neq 0$ when $f'''(x_0)$ exists. (See Problems 9 to 13.)

SECOND-DERIVATIVE TEST. There is a second, and possibly more useful, test for maxima and minima:

1. Solve $f'(x_0) = 0$ for the critical values.
2. For a critical value $x = x_0$:

   $f(x)$ has a maximum value $f(x_0)$ if $f''(x_0) < 0$ (Fig. 13-2(*a*)).

   $f(x)$ has a minimum value $f(x_0)$ if $f''(x_0) > 0$ (Fig. 13-2(*b*)).

   The test fails if $f''(x_0) = 0$ or becomes infinite (Fig. 13-2(*c*) and (*d*)). In this case, the first-derivative test must be used.

(See Problems 14 to 16.)

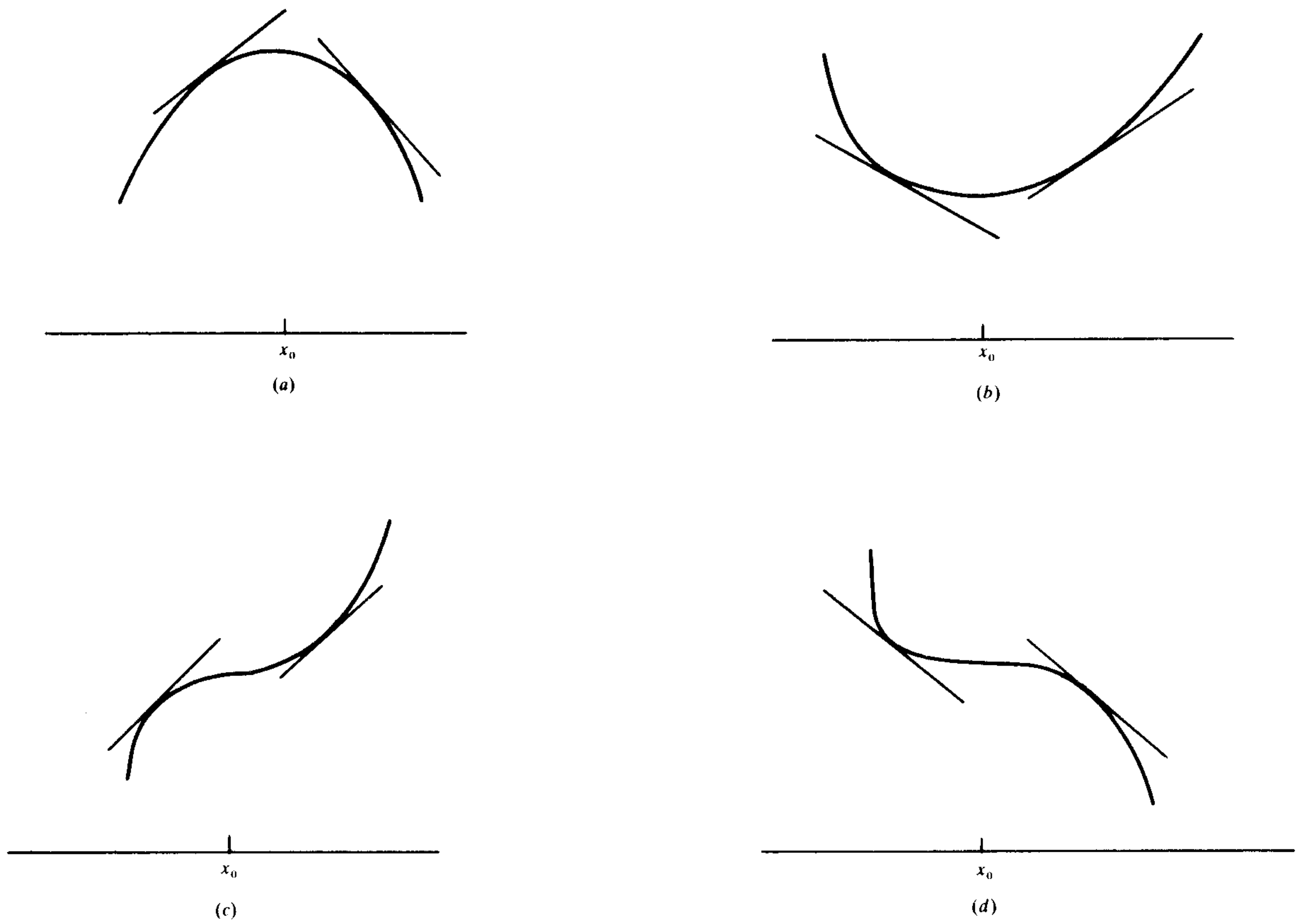


Fig. 13-2

Solved Problems

1. Locate the maximum or minimum values of (*a*) $y = -x^2$; (*b*) $y = (x-3)^2$; (*c*) $y = \sqrt{25 - 4x^2}$; and (*d*) $y = \sqrt{x-4}$.

(*a*) $y = -x^2$ has a relative maximum value $(=0)$ when $x = 0$, since $y = 0$ when $x = 0$ and $y < 0$ when $x \neq 0$.

(*b*) $y = (x-3)^2$ has a relative minimum value $(=0)$ when $x = 3$, since $y = 0$ when $x = 3$ and $y > 0$ when $x \neq 3$.

(*c*) $y = \sqrt{25 - 4x^2}$ has a relative maximum value $(=5)$ when $x = 0$, since $y = 5$ when $x = 0$ and $y < 5$ when $-1 < x < 1$.

(*d*) $y = \sqrt{x-4}$ has neither a relative maximum nor a relative minimum value. (Some authors define relative maximum (minimum) values so that this function has a relative minimum at $x = 4$. See Problem 30.)

2. Given $y = \frac{1}{3}x^2 + \frac{1}{2}x^2 - 6x + 8$, find (*a*) the critical points; (*b*) the intervals on which y is increasing and decreasing; and (*c*) the maximum and minimum values of y.

(*a*) $y' = x^2 + x - 6 = (x+3)(x-2)$. Setting $y' = 0$ gives the critical values $x = -3$ and 2. The critical points are $(-3, \frac{43}{2})$ and $(2, \frac{2}{3})$.

(*b*) When y' is positive, y increases; when y' is negative, y decreases.

When $x < -3$, say $x = -4$, $y' = (-)(-) = +$, and y is increasing.
When $-3 < x < 2$, say $x = 0$, $y' = (+)(-) = -$, and y is decreasing.
When $x > 2$, say $x = 3$, $y' = (+)(+) = +$, and y is increasing.

These results are illustrated by the following diagram (see Fig. 13-3):

$x < -3$	$x = -3$	$-3 < x < 2$	$x = 2$	$x > 2$
$y' = +$ y increases		$y' = -$ y decreases		$y' = +$ y increases

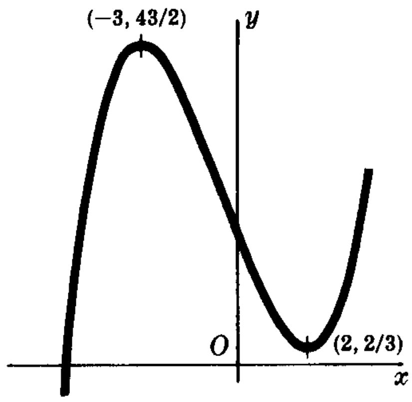


Fig. 13-3

(*c*) We test the critical values $x = -3$ and 2 for maxima and minima:

As x increases through -3, y' changes sign from + to −; hence at $x = -3$, y has a maximum value $\frac{43}{2}$.

As x increases through 2, y' changes sign from − to +; hence at $x = 2$, y has a minimum value $\frac{2}{3}$.

3. Given $y = x^4 + 2x^3 - 3x^2 - 4x + 4$, find (*a*) the intervals on which y is increasing and decreasing, and (*b*) the maximum and minimum values of y.

We have $y' = 4x^3 + 6x^2 - 6x - 4 = 2(x + 2)(2x + 1)(x - 1)$. Setting $y' = 0$ gives the critical values $x = -2, -\frac{1}{2}$, and 1. (See Fig. 13-4.)

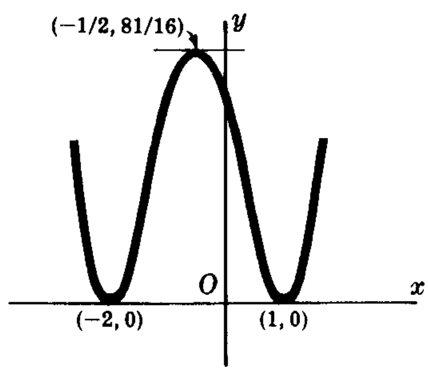


Fig. 13-4

(*a*) When $x < -2$, $y' = 2(-)(-)(-) = -$, and y is decreasing.
When $-2 < x < -\frac{1}{2}$, $y' = 2(+)(-)(-) = +$, and y is increasing.
When $-\frac{1}{2} < x < 1$, $y' = 2(+)(+)(-) = -$, and y is decreasing.
When $x > 1$, $y' = 2(+)(+)(+) = +$, and y is increasing.

These results are illustrated by the following diagram (see Fig. 13-4):

$x<-2$	$x=-2$	$-2<x<-\frac{1}{2}$	$x=-\frac{1}{2}$	$-\frac{1}{2}<x<1$	$x=1$	$x>1$
$y'=-$ y decreases		$y'=+$ y increases		$y'=-$ y decreases		$y'=+$ y increases

(*b*) We test the critical values $x=-2$, $-\frac{1}{2}$, and 1 for maxima and minima:

As x increases through -2, y' changes from $-$ to $+$; hence at $x=-2$, y has a minimum value 0.

As x increases through $-\frac{1}{2}$, y' changes from $+$ to $-$; hence at $x=-\frac{1}{2}$, y has a maximum value 81/16.

As x increases through 1, y' changes from $-$ to $+$; hence at $x=1$, y has a minimum value 0.

4. Show that the curve $y=x^3-8$ has no maximum or minimum value.

Setting $y'=3x^2=0$ gives the critical value $x=0$. But $y'>0$ when $x<0$ and when $x>0$. Hence y has no maximum or minimum value.

The curve has a point of inflection at $x=0$.

5. Examine $y=f(x)=\dfrac{1}{x-2}$ for maxima and minima, and locate the intervals on which the function is increasing and decreasing.

$f'(x)=-\dfrac{1}{(x-2)^2}$. Since $f(2)$ is not defined (that is, $f(x)$ becomes infinite as x approaches 2), there is no critical value. However, $x=2$ may be employed to locate intervals on which $f(x)$ is increasing and decreasing.

$f'(x)<0$ for all $x\neq 2$. Hence $f(x)$ is decreasing on the intervals $x<2$ and $x>2$. (See Fig. 13-5.)

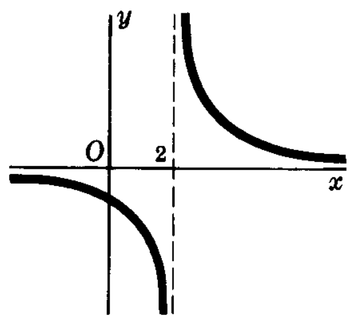


Fig. 13-5

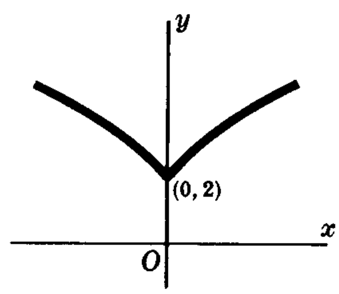


Fig. 13-6

6. Locate the maximum and minimum values of $f(x)=2+x^{2/3}$ and the intervals on which the function is increasing and decreasing.

$f'(x)=\dfrac{2}{3x^{1/3}}$. The critical value is $x=0$, since $f'(x)$ becomes infinite as x approaches 0.

When $x<0$, $f'(x)=-$, and $f(x)$ is decreasing. When $x>0$, $f'(x)=+$, and $f(x)$ is increasing.

Hence, at $x=0$ the function has the minimum value 2. (See Fig. 13-6.)

7. Examine $y=x^{4/3}(1-x)^{1/3}$ for maximum and minimum values.

Here $y'=\dfrac{x^{1/3}(4-5x)}{3(1-x)^{2/3}}$ and the critical values are $x=0$, $\frac{4}{5}$, and 1.

When $x<0$, $y'<0$. When $0<x<\frac{4}{5}$, $y'>0$. When $\frac{4}{5}<x<1$, $y'<0$. When $x>1$, $y'<0$.

The function has a minimum value ($=0$) when $x=0$ and a maximum value ($=\frac{4}{25}\sqrt[3]{20}$) when $x=\frac{4}{5}$.

8. Examine $y=|x|$ for maximum and minimum values.

The function is everywhere defined and has a derivative for all x except $x = 0$. (See Problem 11 of Chapter 9.) Thus, $x = 0$ is a critical value. For $x < 0, f'(x) = -1$; for $x > 0, f'(x) = +1$. The function has a minimum (=0) when $x = 0$. This result is immediate from a figure.

9. Examine $y = 3x^4 - 10x^3 - 12x^2 + 12x - 7$ for concavity and points of inflection.

We have

$$y' = 12x^3 - 30x^2 - 24x + 12$$
$$y'' = 36x^2 - 60x - 24 = 12(3x + 1)(x - 2)$$

Set $y'' = 0$ and solve to obtain the possible points of inflection $x = -\frac{1}{3}$ and 2. Then:

When $x < -\frac{1}{3}$, $y'' = +$, and the arc is concave upward.
When $-\frac{1}{3} < x < 2$, $y'' = -$, and the arc is concave downward.
When $x > 2$, $y'' = +$, and the arc is concave upward.

The points of inflection are $(-\frac{1}{3}, -\frac{322}{27})$ and $(2, -63)$, since y'' changes sign at $x = -\frac{1}{3}$ and $x = 2$ (see Fig. 13-7).

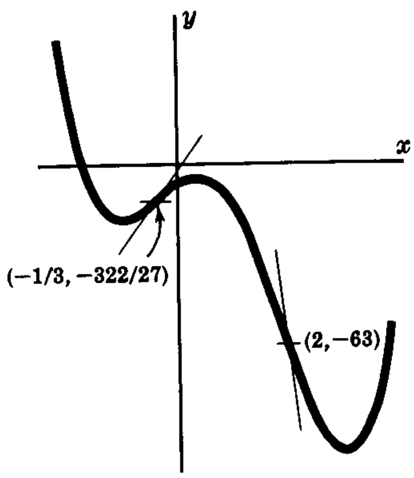


Fig. 13-7

10. Examine $y = x^4 - 6x + 2$ for concavity and points of inflection. (See Fig. 13-8.)

We have $y'' = 12x^2$. The possible point of inflection is at $x = 0$.

On the intervals $x < 0$ and $x > 0$, $y'' = +$, and the arcs on both sides of $x = 0$ are concave upward. The point $(0, 2)$ is not a point of inflection.

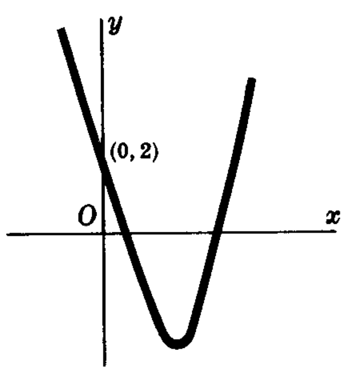


Fig. 13-8

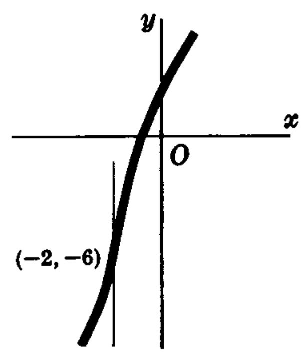


Fig. 13-9

11. Examine $y = 3x + (x + 2)^{3/5}$ for concavity and points of inflection. (See Fig. 13-9.)

Here $$y' = 3 + \frac{3}{5(x+2)^{2/5}} \quad \text{and} \quad y'' = \frac{-6}{25(x+2)^{7/5}}$$

The possible point of inflection is at $x = -2$.

When $x > -2$, $y'' = -$ and the arc is concave downward. When $x < -2$, $y'' = +$ and the arc is concave upward. Hence, $(-2, -6)$ is a point of inflection.

12. Find the equations of the tangents at the points of inflection of $y = f(x) = x^4 - 6x^3 + 12x^2 - 8x$.

A point of inflection exists at $x = x_0$ when $f''(x_0) = 0$ and $f'''(x_0) \neq 0$. Here,

$$f'(x) = 4x^3 - 18x^2 + 24x - 8$$
$$f''(x) = 12x^2 - 36x + 24 = 12(x-1)(x-2)$$
$$f'''(x) = 24x - 36 = 12(2x-3)$$

The possible points of inflection are at $x = 1$ and 2. Since $f'''(1) \neq 0$ and $f'''(2) \neq 0$, the points $(1, -1)$ and $(2, 0)$ are points of inflection.

At $(1, -1)$, the slope of the tangent is $m = f'(1) = 2$, and its equation is

$$y - y_1 = m(x - x_1) \quad \text{or} \quad y + 1 = 2(x-1) \quad \text{or} \quad y = 2x - 3$$

At $(2, 0)$, the slope is $f'(2) = 0$, and the equation of the tangent is $y = 0$.

13. Show that the points of inflection of $y = \dfrac{a-x}{x^2+a^2}$ lie on a straight line, and find its equation.

Here $$y' = \frac{x^2 - 2ax - a^2}{(x^2+a^2)^2} \quad \text{and} \quad y'' = -2\,\frac{x^3 - 3ax^2 - 3a^2x + a^3}{(x^2+a^2)^3}$$

Now $x^3 - 3ax^2 - 3a^2x + a^3 = 0$ when $x = -a$ and $a(2 \pm \sqrt{3})$; hence the points of inflection are $(-a, 1/a)$, $(a(2+\sqrt{3}), (1-\sqrt{3})/4a)$, and $(a(2-\sqrt{3}), (1+\sqrt{3})/4a)$. The slope of the line joining any two of these points is $-1/4a^2$, and the equation of the line of inflection points is $x + 4a^2y = 3a$.

14. Examine $f(x) = x(12-2x)^2$ for maxima and minima using the second-derivative method.

Here $f'(x) = 12(x^2 - 8x + 12) = 12(x-2)(x-6)$. Hence, the critical values are $x = 2$ and 6.

Also, $f''(x) = 12(2x-8) = 24(x-4)$. Because $f''(2) < 0$, $f(x)$ has a maximum value $(=128)$ at $x = 2$. Because $f''(6) > 0$, $f(x)$ has a minimum value $(=0)$ at $x = 6$.

15. Examine $y = x^2 + 250/x$ for maxima and minima using the second-derivative method.

Here $y' = 2x - \dfrac{250}{x^2} = \dfrac{2(x^3-125)}{x^2}$, so the critical value is $x = 5$.

Also, $y'' = 2 + \dfrac{500}{x^3}$. Because $y'' > 0$ at $x = 5$, y has a minimum value $(=75)$ at $x = 5$.

16. Examine $y = (x-2)^{2/3}$ for maximum and minimum values.

$y' = \dfrac{2}{3}(x-2)^{-1/3} = \dfrac{2}{3(x-2)^{1/3}}$. Hence, the critical value is $x = 2$.

$y'' = -\dfrac{2}{9}(x-2)^{-4/3} = -\dfrac{2}{9(x-2)^{4/3}}$ becomes infinite as x approaches 2. Hence the second-derivative test fails, and we employ the first-derivative method: When $x < 2$, $y' = -$; when $x > 2$, $y' = +$. Hence y has a relative minimum $(=0)$ at $x = 2$.

17. A function $f(x)$ is said to be increasing at $x = x_0$ if for $h > 0$ and sufficiently small, $f(x_0 - h) < f(x_0) < f(x_0 + h)$. Prove: If $f'(x_0) > 0$, then $f(x)$ is increasing at $x = x_0$.

Since $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) > 0$, we have $\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} > 0$ for sufficiently small $|\Delta x|$ by Problem 4 of Chapter 8.

If $\Delta x < 0$, then $f(x_0 + \Delta x) - f(x_0) < 0$, and setting $\Delta x = -h$ yields $f(x_0 - h) < f(x_0)$. If $\Delta x > 0$, say $\Delta x = h$, then $f(x_0 + h) > f(x_0)$. Hence, $f(x_0 - h) < f(x_0) < f(x_0 + h)$ as required in the definition. (See Problem 33 for a companion theorem.)

18. Prove: If $y = f(x)$ is differentiable on $a \le x \le b$ and $f(x)$ has a relative maximum at $x = x_0$, where $a < x_0 < b$, then $f'(x_0) = 0$.

Since $f(x)$ has a relative maximum at $x = x_0$, for every Δx with $|\Delta x|$ sufficiently small we have

$$f(x_0 + \Delta x) < f(x_0); \qquad \text{so} \qquad f(x_0 + \Delta x) - f(x_0) < 0$$

When $\Delta x < 0$,

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} > 0 \qquad \text{and} \qquad f'(x_0) = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \ge 0$$

When $\Delta x > 0$,

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} < 0 \qquad \text{and} \qquad f'(x_0) = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \le 0$$

Thus, $0 \le f'(x_0) \le 0$ and $f'(x_0) = 0$, as was to be proved. (See Problem 34 for a companion theorem.)

19. Prove the second-derivative test for maximum and minimum: If $f(x)$ and $f'(x)$ are differentiable on $a \le x \le b$, if $x = x_0$ (where $a < x_0 < b$) is a critical value for $f(x)$, and if $f''(x_0) > 0$, then $f(x)$ has a relative minimum value at $x = x_0$.

Since $f''(x_0) > 0, f'(x)$ is increasing at $x = x_0$ and there exists an $h > 0$ such that $f'(x_0 - h) < f'(x_0) < f'(x_0 + h)$. Thus, when x is near to but less than $x_0, f'(x) < f'(x_0)$; when x is near to but greater than x_0, $f'(x) > f'(x_0)$. Now since $f'(x_0) = 0$, $f'(x) < 0$ when $x < x_0$ and $f'(x) > 0$ when $x > x_0$. By the First-Derivative Test, $f(x)$ has a relative minimum at $x = x_0$. (It is left for the reader to consider the companion theorem for relative maximum.)

20. Consider the problem of locating the point (X, Y) on the hyperbola $x^2 - y^2 = 1$ nearest a given point $P(a, 0)$, where $a > 0$. We have $D^2 = (X - a)^2 + Y^2$ for the square of the distance between the two points and $X^2 - Y^2 = 1$, since (X, Y) is on the hyperbola.

Expressing D^2 as a function of X alone, we obtain

$$f(X) = (X - a)^2 + X^2 - 1 = 2X^2 - 2aX + a^2 - 1$$

with critical value $X = \frac{1}{2}a$.

Take $a = \frac{1}{2}$. No point is found, since Y is imaginary for the critical value $X = \frac{1}{4}$. From a figure, however, it is clear that the point on the hyperbola nearest $P(\frac{1}{4}, 0)$ is $V(1, 0)$. The trouble here is that we have overlooked the fact that $f(X) = (X - \frac{1}{2})^2 + X^2 - 1$ is to be minimized subject to the restriction $X \ge 1$. (Note that this restriction does not arise from $f(X)$ itself. The function $f(X)$, with X unrestricted, has indeed a relative minimum at $X = \frac{1}{4}$.) On the interval $X \ge 1$, $f(X)$ has an absolute minimum at the endpoint $X = 1$, but no relative minimum. It is left as an exercise to examine the cases $a = \sqrt{2}$ and $a = 3$.

Supplementary Problems

21. Examine each function of Problem 1 and determine the intervals on which it is increasing and decreasing.

Ans. (*a*) increasing $x<0$, decreasing $x>0$; (*b*) increasing $x>3$, decreasing $x<3$; (*c*) increasing $-\frac{5}{2}<x<0$, decreasing $0<x<\frac{5}{2}$; (*d*) increasing $x>4$

22. (*a*) Show that $y = x^5 + 20x - 6$ is an increasing function for all values of x.
(*b*) Show that $y = 1 - x^3 - x^7$ is a decreasing function for all values of x.

23. Examine each of the following for relative maximum and minimum values, using the first-derivative test.

(*a*) $f(x) = x^2 + 2x - 3$ *Ans.* $x=-1$ yields relative minimum -4

(*b*) $f(x) = 3 + 2x - x^2$ *Ans.* $x=1$ yields relative maximum 4

(*c*) $f(x) = x^3 + 2x^2 - 4x - 8$ *Ans.* $x=\frac{2}{3}$ yields relative minimum $-\frac{256}{27}$; $x=-2$ yields relative maximum 0

(*d*) $f(x) = x^3 - 6x^2 + 9x - 8$ *Ans.* $x=1$ yields relative maximum -4; $x=3$ yields relative minimum -8

(*e*) $f(x) = (2-x)^3$ *Ans.* neither relative maximum nor relative minimum

(*f*) $f(x) = (x^2-4)^2$ *Ans.* $x=0$ yields relative maximum 16; $x=\pm 2$ yields relative minimum 0

(*g*) $f(x) = (x-4)^4(x+3)^3$ *Ans.* $x=0$ yields relative maximum 6912; $x=4$ yields relative minimum 0; $x=-3$ yields neither

(*h*) $f(x) = x^3 + 48/x$ *Ans.* $x=-2$ yields relative maximum -32; $x=2$ yields relative minimum 32

(*i*) $f(x) = (x-1)^{1/3}(x+2)^{2/3}$ *Ans.* $x=-2$ yields relative maximum 0; $x=0$ yields relative minimum $-\sqrt[3]{4}$; $x=1$ yields neither

24. Examine the functions of Problem 23(*a*) to (*f*) for relative maximum and minimum values using the second-derivative method. Also determine the points of inflection and the intervals on which the curve is concave upward and concave downward.

Ans. (*a*) no inflection point, concave upward everywhere
(*b*) no inflection point, concave downward everywhere
(*c*) inflection point $x=-\frac{2}{3}$; concave up for $x>-\frac{2}{3}$, concave down for $x<-\frac{2}{3}$
(*d*) inflection point $x=2$; concave up for $x>2$, concave down for $x<2$
(*e*) inflection point $x=2$; concave down for $x>2$, concave up for $x<2$
(*f*) inflection point $x=\pm 2\sqrt{3}/3$; concave up for $x>2\sqrt{3}/3$ and $x<-2\sqrt{3}/3$, concave down for $-2\sqrt{3}/3<x<2\sqrt{3}/3$

25. Show that $y = \dfrac{ax+b}{cx+d}$ has neither a relative maximum nor a relative minimum.

26. Examine $y = x^3 - 3px + q$ for relative maximum and minimum values.

Ans. minimum $= q - 2p^{3/2}$, maximum $= q + 2p^{3/2}$ if $p>0$; otherwise neither.

27. Show that $y = (a_1 - x)^2 + (a_2 - x)^2 + \cdots + (a_n - x)^2$ has a relative minimum when $x = (a_1 + a_2 + \cdots + a_n)/n$.

28. Prove: If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then there is a point of inflection at $x = x_0$.

29. Prove: If $y = ax^3 + bx^2 + cx + d$ has two critical points, they are bisected by the point of inflection. If the curve has just one critical point, it is the point of inflection.

30. A function $f(x)$ is said to have an absolute maximum (minimum) value at $x = x_0$ provided $f(x_0)$ is greater (less) than or equal to every other value of the function on its domain of definition. Use graphs to verify: (*a*) $y = -x^2$ has an absolute maximum at $x=0$; (*b*) $y = (x-3)^2$ has an absolute minimum $(=0)$ at $x=3$; (*c*) $y = \sqrt{25-4x^2}$ has an absolute maximum $(=5)$ at $x=0$ and an absolute minimum $(=0)$ at $x=\pm 5/2$; (*d*) $y = \sqrt{x-4}$ has an absolute minimum $(=0)$ at $x=4$.

31. Examine the following for absolute maximum and minimum values on the given interval only:

(a) $y = -x^2$ on $-2 < x < 2$ *Ans.* maximum $(=0)$ at $x = 0$

(b) $y = (x-3)^2$ on $0 \le x \le 4$ *Ans.* maximum $(=9)$ at $x = 0$; minimum $(=0)$ at $x = 3$

(c) $y = \sqrt{25 - 4x^2}$ on $-2 \le x \le 2$ *Ans.* maximum $(=5)$ at $x = 0$; minimum $(=3)$ at $x = \pm 2$

(d) $y = \sqrt{x-4}$ on $4 \le x \le 29$ *Ans.* maximum $(=5)$ at $x = 29$; minimum $(=0)$ at $x = 4$

Note: These are the greatest and least values of Property 8.2 for continuous functions.

32. Verify: A function $f(x)$ is increasing (decreasing) at $x = x_0$ if the angle of inclination of the tangent at $x = x_0$ to the curve $y = f(x)$ is acute (obtuse).

33. Prove the companion theorem of Problem 17 for a decreasing function: If $f'(x_0) < 0$, then $f(x)$ is decreasing at x_0.

34. State and prove the companion theorem of Problem 18 for a relative minimum: If $y = f(x)$ is differentiable on $a \le x \le b$ and $f(x)$ has a relative minimum at $x = x_0$, where $a < x_0 < b$, then $f'(x_0) = 0$.

35. Examine $2x^2 - 4xy + 3y^2 - 8x + 8y - 1 = 0$ for maximum and minimum points.

Ans. maximum at $(5, 3)$; minimum at $(-1, -3)$

36. An electric current, when flowing in a circular coil of radius r, exerts a force $F = \dfrac{kx}{(x^2 + r^2)^{5/2}}$ on a small magnet located a distance x above the center of the coil. Show that F is maximum when $x = \frac{1}{2}r$.

37. The work done by a voltaic cell of constant electromotive force E and constant internal resistance r in passing a steady current through an external resistance R is proportional to $E^2R/(r + R)^2$. Show that the work done is maximum when $R = r$.

Chapter 14

Applied Problems Involving Maxima and Minima

PROBLEMS INVOLVING MAXIMA AND MINIMA. In simpler applications, it is rarely necessary to rigorously prove that a certain critical value yields a relative maximum or minimum. The correct determination can usually be made by virtue of an intuitive understanding of the problem. However, it is generally easy to justify such a determination with the first-derivative test or the second-derivative test.

A relative maximum or minimum may also be an *absolute* maximum or minimum (that is, the greatest or smallest value) of a function. For a continuous function $f(x)$ on a closed interval $[a, b]$, there must exist an absolute maximum and an absolute minimum, and a systematic procedure for finding them is available. Find all the critical values $c_1, c_2, \ldots, c_n$ for the function in $[a, b]$, and then calculate $f(x)$ for each of the arguments $c_1, c_2, \ldots, c_n$, and for the endpoints a and b. The largest of these values is the absolute maximum, and the least of these values is the absolute minimum, of the function on $[a, b]$.

Solved Problems

1. Divide the number 120 into two parts such that the product P of one part and the square of the other is a maximum.

Let x be one part, and $120 - x$ the other part. Then $P = (120 - x)x^2$, and $0 \le x \le 120$.

Since $dP/dx = 3x(80 - x)$, the critical values are $x = 0$ and $x = 80$. Now $P(0) = 0$, $P(80) = 256{,}000$, and $P(120) = 0$; hence the maximum value of P occurs when $x = 80$. The required parts are 80 and 40.

2. A sheet of paper for a poster is to be 18 ft^2 in area. The margins at the top and bottom are to be 9 in wide, and at the sides 6 in. What should be the dimensions of the sheet to maximize the printed area?

Let x be one dimension of the sheet, in feet. Then $18/x$ is the other dimension. (See Fig. 14-1.) The only restriction on x is that $x > 0$. The printed area (in square feet) is $A = (x - 1)\left(\frac{18}{x} - \frac{3}{2}\right)$, and $\frac{dA}{dx} = \frac{18}{x^2} - \frac{3}{2}$.

Solving $dA/dx = 0$ yields the critical value $x = 2\sqrt{3}$. Since $\frac{d^2A}{dx^2} = -\frac{36}{x^3}$ is negative when $x = 2\sqrt{3}$, the second-derivative test tells us that A has a relative maximum at that value. Since $2\sqrt{3}$ is the *only* critical value, A must achieve an *absolute* maximum at $x = 2\sqrt{3}$. (Why?) Thus, one side is $2\sqrt{3}$ ft, and the other is $18/(2\sqrt{3}) = 3\sqrt{3}$ ft.

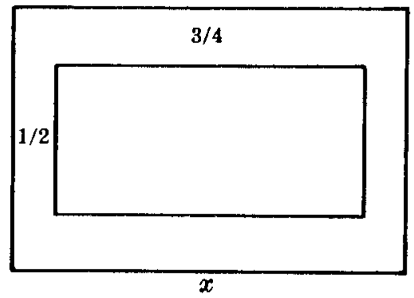


Fig. 14-1

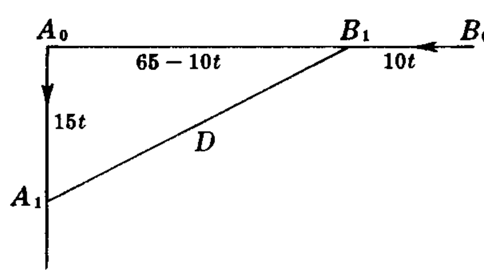


Fig. 14-2

3. At 9 A.M. ship B is 65 mi due east of another ship A. Ship B is then sailing due west at 10 mi/h, and A is sailing due south at 15 mi/h. If they continue on their respective courses, when will they be nearest one another, and how near? (See Fig. 14-2.)

Let A_0 and B_0 be the positions of the ships at 9 A.M., and A_1 and B_1 be their positions t hours later. The distance covered in t hours by A is $15t$ miles; by B, $10t$ miles.

The distance D between the ships is given by $D^2 = (15t)^2 + (65 - 10t)^2$. Then $\dfrac{dD}{dt} = \dfrac{325t - 650}{D}$. Solving $\dfrac{dD}{dt} = 0$ gives the critical value $t = 2$. Since $D > 0$ and $325t - 650$ is positive to the right of $t = 2$ and negative to the left of $t = 2$, the first-derivative test tells us that $t = 2$ yields a relative minimum for D. Since $t = 2$ is the only critical value, that relative minimum is an absolute minimum.

Putting $t = 2$ in, $D^2 = (15t)^2 + (65 - 10t)^2$ gives $D = 15\sqrt{13}$ mi. Hence, the ships are nearest at 11 A.M., at which time they are $15\sqrt{13}$ mi apart.

4. A cylindrical container with circular base is to hold 64 in^3. Find its dimensions so that the amount (surface area) of metal required is a minimum when the container is (*a*) an open cup and (*b*) a closed can.

Let r and h be, respectively, the radius of the base and the height in inches, A the amount of metal, and V the volume of the container.

(*a*) Here $V = \pi r^2 h = 64$, and $A = 2\pi rh + \pi r^2$. To express A as a function of one variable, we solve for h in the first relation (because it is easier) and substitute in the second, obtaining

$$A = 2\pi r \frac{64}{\pi r^2} + \pi r^2 = \frac{128}{r} + \pi r^2 \qquad \text{and} \qquad \frac{dA}{dr} = -\frac{128}{r^2} + 2\pi r = \frac{2(\pi r^3 - 64)}{r^2}$$

and the critical value is $r = 4/\sqrt[3]{\pi}$. Then $h = 64/\pi r^2 = 4/\sqrt[3]{\pi}$. Thus, $r = h = 4/\sqrt[3]{\pi}$ in.

Now $dA/dr > 0$ to the right of the critical value, and $dA/dr < 0$ to the left of the critical value. So, by the first-derivative test, we have a relative minimum. Since there is no other critical value, that relative minimum is an absolute minimum.

(*b*) Here again $V = \pi r^2 h = 64$, but $A = 2\pi rh + 2\pi r^2 = 2\pi r(64/\pi r^2) + 2\pi r^2 = 128/r + 2\pi r^2$. Hence,

$$\frac{dA}{dr} = -\frac{128}{r^2} + 4\pi r = \frac{4(\pi r^3 - 32)}{r^2}$$

and the critical value is $r = 2\sqrt[3]{4/\pi}$. Then $h = 64/\pi r^2 = 4\sqrt[3]{4/\pi}$. Thus, $h = 2r = 4\sqrt[3]{4/\pi}$ in. That we have found an absolute minimum can be shown as in part (*a*).

5. The total cost of producing x radio sets per day is $\$(\frac{1}{4}x^2 + 35x + 25)$, and the price per set at which they may be sold is $\$(50 - \frac{1}{2}x)$.

(*a*) What should be the daily output to obtain a maximum total profit?

(*b*) Show that the cost of producing a set is a relative minimum at that output.

(*a*) The profit on the sale of x sets per day is $P = x(50 - \frac{1}{2}x) - (\frac{1}{4}x^2 + 35x + 25)$. Then $\dfrac{dP}{dx} = 15 - \dfrac{3x}{2}$; solving $dP/dx = 0$ gives the critical value $x = 10$.

Since $d^2P/dx^2 = -\frac{3}{2} < 0$, the second-derivative test shows that we have found a relative maximum. Since $x = 10$ is the only critical value, the relative maximum is an absolute maximum. Thus, the daily output that maximizes profit is 10 sets per day.

(*b*) The cost of producing a set is $C = \dfrac{\frac{1}{4}x^2 + 35x + 25}{x} = \dfrac{1}{4}x + 35 + \dfrac{25}{x}$. Then $\dfrac{dC}{dx} = \dfrac{1}{4} - \dfrac{25}{x^2}$; solving $dC/dx = 0$ gives the critical value $x = 10$.

Since $\dfrac{d^2C}{dx^2} = \dfrac{50}{x^3} > 0$ when $x = 10$, we have found a relative minimum. Since there is only one critical value, this must be an absolute minimum.

6. The cost of fuel to run a locomotive is proportional to the square of the speed and is \$25/h for a speed of 25 mi/h. Other costs amount to \$100/h, regardless of the speed. Find the speed that minimizes the cost per mile.

Let v = required speed, and C = total cost per mile. The fuel cost *per hour* is kv^2, where the constant k is to be determined. When $v = 25$ mi/h, $kv^2 = 625k = 25$; hence $k = \frac{1}{25}$.

$$C \text{ (in \$/mi)} = \frac{\text{cost in \$/h}}{\text{speed in mi/h}} = \frac{v^2/25 + 100}{v} = \frac{v}{25} + \frac{100}{v}$$

and $\dfrac{dC}{dv} = \dfrac{1}{25} - \dfrac{100}{v^2} = \dfrac{(v-50)(v+50)}{25v^2}$. Since $v > 0$, the only relevant critical value is $v = 50$.

Because d^2C/dv^2 is positive to the right of $v = 50$ and negative to the left of $v = 50$, the first-derivative test tells us that C assumes a relative minimum at $v = 50$. Since $v = 50$ is the only positive critical number, the most economical speed is 50 mi/h.

7. A man in a rowboat at P in Fig. 14-3, 5 mi from the nearest point A on a straight shore, wishes to reach a point B, 6 mi from A along the shore, in the shortest time. Where should he land if he can row 2 mi/h and walk 4 mi/h?

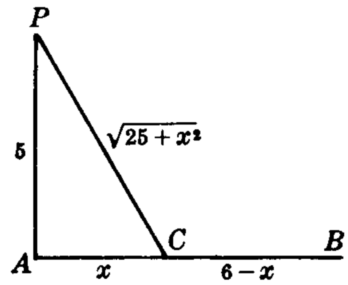


Fig. 14-3

Let C be the point between A and B at which the man lands, and let $AC = x$.

The distance rowed is $PC = \sqrt{25 + x^2}$, and the rowing time required is $t_1 = \dfrac{\text{distance}}{\text{speed}} = \dfrac{\sqrt{25+x^2}}{2}$. The distance walked is $CB = 6 - x$, and the walking time required is $t_2 = (6 - x)/4$. Hence, the total time required is

$$t = t_1 + t_2 = \tfrac{1}{2}\sqrt{25 + x^2} + \tfrac{1}{4}(6 - x) \qquad \text{and} \qquad \frac{dt}{dx} = \frac{x}{2\sqrt{25+x^2}} - \frac{1}{4} = \frac{2x - \sqrt{25+x^2}}{4\sqrt{25+x^2}}$$

The critical value, obtained from $2x - \sqrt{25 + x^2} = 0$, is $x = \frac{5}{3}\sqrt{3} \sim 2.89$. Thus, he should land at a point 2.89 mi from A toward B. (How do we know that this point yields the *shortest* time?)

8. A given rectangular area is to be fenced off in a field that lies along a straight river. If no fencing is needed along the river, show that the least amount of fencing will be required when the length of the field is twice its width.

Let x be the length of the field, and y its width. The area of the field is $A = xy$. The fencing required is $F = x + 2y$, and $\dfrac{dF}{dx} = 1 + 2\dfrac{dy}{dx}$. When $\dfrac{dF}{dx} = 0$, $\dfrac{dy}{dx} = -\frac{1}{2}$.

Also, $\dfrac{dA}{dx} = 0 = y + x\dfrac{dy}{dx}$. Then $y - \frac{1}{2}x = 0$, and $x = 2y$ as required.

To see that F has been minimized, note that $\dfrac{dy}{dx} = -\dfrac{y^2}{A}$ and

$$\frac{d^2F}{dx^2} = 2\frac{d^2y}{dx^2} = 2\left(-2\frac{y}{A}\frac{dy}{dx}\right) = -4\frac{y}{A}\left(-\frac{1}{2}\right) = 2\frac{y}{A} > 0 \qquad \text{when} \qquad \frac{dy}{dx} = -\frac{1}{2}$$

Now use the second-derivative test and the uniqueness of the critical value.

9. Find the dimensions of the right circular cone of minimum volume V that can be circumscribed about a sphere of radius 8 in.

Let x = radius of base of cone, and $y + 8$ = altitude of cone (Fig. 14-4). From similar right triangles ABC and AED, we have

$$\frac{x}{8} = \frac{y+8}{\sqrt{y^2-64}} \quad \text{or} \quad x^2 = \frac{64(y+8)^2}{y^2-64} = \frac{64(y+8)}{y-8}$$

Also,

$$V = \frac{(\pi x^2)(y+8)}{3} = \frac{64\pi(y+8)^2}{3(y-8)} \quad \text{and} \quad \frac{dV}{dy} = \frac{64\pi(y+8)(y-24)}{3(y-8)^2}$$

The pertinent critical value is $y = 24$. Then the altitude of the cone is $y + 8 = 32$ in, and the radius of the base is $x = 8\sqrt{2}$ in. (How do we know that the volume has been minimized?)

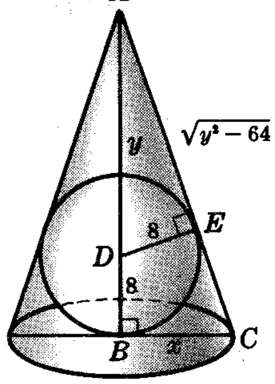


Fig. 14-4

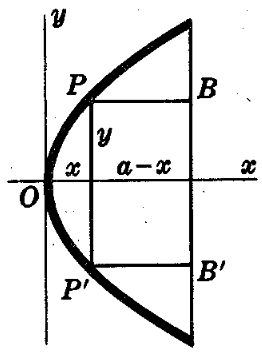


Fig. 14-5

10. Find the dimensions of the rectangle of maximum area A that can be inscribed in the portion of the parabola $y^2 = 4px$ intercepted by the line $x = a$.

Let $PBB'P'$ in Fig. 14-5 be the rectangle, and (x, y) the coordinates of P. Then

$$A = 2y(a-x) = 2y\left(a - \frac{y^2}{4p}\right) = 2ay - \frac{y^3}{2p} \quad \text{and} \quad \frac{dA}{dy} = 2a - \frac{3y^2}{2p}$$

Solving $dA/dy = 0$ yields the critical value $y = \sqrt{4ap/3}$. The dimensions of the rectangle are $2y = \frac{4}{3}\sqrt{3ap}$ and $a - x = a - y^2/4p = 2a/3$.

Since $\dfrac{d^2A}{dy^2} = -\dfrac{3}{p}y < 0$, the second-derivative test and the uniqueness of the critical value ensure that we have found the maximum area.

11. Find the height of the right circular cylinder of maximum volume V that can be inscribed in a sphere of radius R. (See Fig. 14-6.)

Let r be the radius of the base, and $2h$ the height, of the cylinder. From the geometry, $V = 2\pi r^2 h$ and $r^2 + h^2 = R^2$. Then

$$\frac{dV}{dr} = 2\pi\left(r^2\frac{dh}{dr} + 2rh\right) \quad \text{and} \quad 2r + 2h\frac{dh}{dr} = 0$$

From the last relation, $\dfrac{dh}{dr} = -\dfrac{r}{h}$, so $\dfrac{dV}{dr} = 2\pi\left(-\dfrac{r^3}{h} + 2rh\right)$. When V is a maximum, $dV/dr = 0$, from which $r^2 = 2h^2$.

Then $R^2 = r^2 + h^2 = 2h^2 + h^2$, so that $h = R/\sqrt{3}$ and the height of the cylinder is $2h = 2R/\sqrt{3}$. The second-derivative test can be used to verify that we have found a maximum value of V.

12. A wall of a building is to be braced by a beam which must pass over a parallel wall 10 ft high and 8 ft from the building. Find the length L of the shortest beam that can be used.

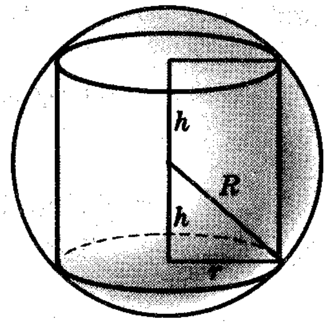


Fig. 14-6

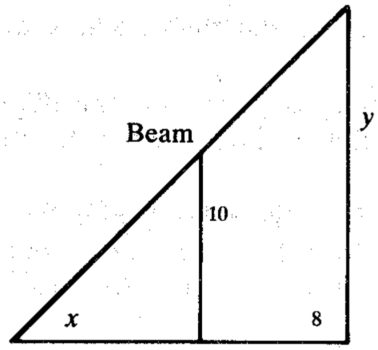


Fig. 14-7

Let x be the distance from the foot of the beam to the foot of the parallel wall, and y the distance from the ground to the top of the beam, in feet. (See Fig. 14-7.) Then $L = \sqrt{(x+8)^2 + y^2}$. Also, from similar triangles, $\frac{y}{10} = \frac{x+8}{x}$, so $y = \frac{10(x+8)}{x}$. Then

$$L = \sqrt{(x+8)^2 + \frac{100(x+8)^2}{x^2}} = \frac{x+8}{x}\sqrt{x^2+100}$$

$$\frac{dL}{dx} = \frac{x[(x^2+100)^{1/2} + x(x+8)(x^2+100)^{-1/2}] - (x+8)(x^2+100)^{1/2}}{x^2} = \frac{x^3 - 800}{x^2\sqrt{x^2+100}}$$

The relevant critical value is $x = 2\sqrt[3]{100}$. The length of the shortest beam is

$$\frac{2\sqrt[3]{100}+8}{2\sqrt[3]{100}}\sqrt{4\sqrt[3]{10{,}000}+100} = (\sqrt[3]{100}+4)^{3/2} \text{ ft}$$

The first-derivative test guarantees that we really have found the shortest length.

Supplementary Problems

13. The sum of two positive numbers is 20. Find the numbers (*a*) if their product is a maximum; (*b*) if the sum of their squares is a minimum; (*c*) if the product of the square of one and the cube of the other is a maximum. *Ans.* (*a*) 10, 10; (*b*) 10, 10; (*c*) 8, 12

14. The product of two positive number is 16. Find the numbers (*a*) if their sum is least; (*b*) if the sum of one and the square of the other is least. *Ans.* (*a*) 4, 4; (*b*) 8, 2

15. An open rectangular box with square ends is to be built to hold 6400 ft^3 at a cost of $\$0.75/\text{ft}^2$ for the base and $\$0.25/\text{ft}^2$ for the sides. Find the most economical dimensions. *Ans.* $20 \times 20 \times 16$ ft

16. A wall 8 ft high is $3\frac{3}{8}$ ft from a house. Find the shortest ladder that will reach from the ground to the house when leaning over the wall. *Ans.* $15\frac{5}{8}$ ft

17. A company offers the following schedule of charges: \$30 per thousand for orders of 50,000 or less, with the charge per thousand decreased by $37\frac{1}{2}$¢ for each thousand above 50,000. Find the order size that makes the company's receipts a maximum. *Ans.* 65,000

18. Find the equation of the line through the point (3, 4) which cuts from the first quadrant a triangle of minimum area. *Ans.* $4x + 3y - 24 = 0$

19. At what first-quadrant point on the parabola $y = 4 - x^2$ does the tangent, together with the coordinate axes, determine a triangle of minimum area. *Ans.* $(2\sqrt{3}/3, 8/3)$

20. Find the minimum distance from the point $(4, 2)$ to the parabola $y^2 = 8x$. *Ans.* $2\sqrt{2}$ units

21. A tangent is drawn to the ellipse $x^2/25 + y^2/16 = 1$ so that the part intercepted by the coordinate axes is a minimum. Show that its length is 9 units.

22. A rectangle is inscribed in the ellipse $x^2/400 + y^2/225 = 1$ with its sides parallel to the axes of the ellipse. Find the dimensions of the rectangle of (*a*) maximum area and (*b*) maximum perimeter which can be so inscribed. *Ans.* (*a*) $20\sqrt{2} \times 15\sqrt{2}$; (*b*) 32×18

23. Find the radius R of the right circular cone of maximum volume that can be inscribed in a sphere of radius r. *Ans.* $R = \frac{2}{3}r\sqrt{2}$

24. A right circular cylinder is inscribed in a right circular cone of radius r. Find the radius R of the cylinder (*a*) if its volume is a maximum; (*b*) if its lateral area is a maximum.

Ans. (*a*) $R = \frac{2}{3}r$; (*b*) $R = \frac{1}{2}r$

25. Show that a conical tent of given capacity will require the least amount of material when its height is $\sqrt{2}$ times the radius of the base.

26. Show that the equilateral triangle of altitude $3r$ is the isosceles triangle of least area circumscribing a circle of radius r.

27. Determine the dimensions of the right circular cylinder of maximum lateral surface that can be inscribed in a sphere of radius 8 in. *Ans.* $h = 2r = 8\sqrt{2}$ in

28. Investigate the possibility of inscribing a right circular cylinder of maximum total area in a right circular cone of radius r and height h. *Ans.* if $h > 2r$, radius of cylinder $= \frac{1}{2}hr/(h - r)$

Chapter 15

Rectilinear and Circular Motion

RECTILINEAR MOTION. The motion of a particle P along a straight line is completely described by the equation $s = f(t)$, where t is time and s is the directed distance of P from a fixed point O in its path.

The *velocity* of P at time t is $v = ds/dt$. If $v > 0$, then P is moving in the direction of increasing s. If $v < 0$, then P is moving in the direction of decreasing s.

The *speed* of P is the absolute value $|v|$ of its velocity.

The *acceleration* of P at time t is $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$. If $a > 0$, then v is increasing; if $a < 0$, then v is decreasing.

If v and a have the same sign, the speed of P is increasing. If v and a have opposite signs, the speed of P is decreasing. (See Problems 1 to 5.)

CIRCULAR MOTION. The motion of a particle P along a circle is completely described by the equation $\theta = f(t)$, where θ is the central angle (in radians) swept over in time t by a line joining P to the center of the circle.

The *angular velocity* of P at time t is $\omega = d\theta/dt$.

The *angular acceleration* of P at time t is $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$.

If $\alpha =$ constant for all t, then P moves with constant angular acceleration. If $\alpha = 0$ for all t, then P moves with constant angular velocity. (See Problem 6.)

Solved Problems

In the following problems on straight-line motion, distance s is in feet and time t is in seconds.

1. A body moves along a straight line according to the law $s = \frac{1}{2}t^3 - 2t$. Determine its velocity and acceleration at the end of 2 seconds.

$v = \dfrac{ds}{dt} = \dfrac{3}{2}t^2 - 2$; hence, when $t = 2$, $v = \frac{3}{2}(2)^2 - 2 = 4$ ft/sec.

$a = \dfrac{dv}{dt} = 3t$; hence, when $t = 2$, $a = 3(2) = 6$ ft/sec^2.

2. The path of a particle moving in a straight line is given by $s = t^3 - 6t^2 + 9t + 4$.

(*a*) Find s and a when $v = 0$.
(*b*) Find s and v when $a = 0$.
(*c*) When is s increasing?
(*d*) When is v increasing?
(*e*) When does the direction of motion change?

We have $\qquad v = \dfrac{ds}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3) \qquad a = \dfrac{dv}{dt} = 6(t-2)$

(*a*) When $v = 0$, $t = 1$ and 3. When $t = 1$, $s = 8$ and $a = -6$. When $t = 3$, $s = 4$ and $a = 6$.
(*b*) When $a = 0$, $t = 2$. At $t = 2$, $s = 6$ and $v = -3$.
(*c*) s is increasing when $v > 0$, that is, when $t < 1$ and $t > 3$.
(*d*) v is increasing when $a > 0$, that is, when $t > 2$.
(*e*) The direction of motion changes when $v = 0$ and $a \neq 0$. From (*a*) the direction changes when $t = 1$ and $t = 3$.

3. A body moves along a horizontal line according to $s = f(t) = t^3 - 9t^2 + 24t$.
(*a*) When is s increasing, and when is it decreasing?
(*b*) When is v increasing, and when is it decreasing?
(*c*) When is the speed of the body increasing, and when is it decreasing?
(*d*) Find the total distance traveled in the first 5 seconds of motion.

We have $$v = \frac{ds}{dt} = 3t^2 - 18t + 24 = 3(t-2)(t-4) \qquad a = \frac{dv}{dt} = 6(t-3)$$

(*a*) s is increasing when $v > 0$, that is, when $t < 2$ and $t > 4$.
s is decreasing when $v < 0$, that is, when $2 < t < 4$.
(*b*) v is increasing when $a > 0$, that is, when $t > 3$.
v is decreasing when $a < 0$, that is, when $t < 3$.
(*c*) The speed is increasing when v and a have the same sign, and decreasing when v and a have opposite signs. Since v may change sign when $t = 2$ and $t = 4$ while a may change sign at $t = 3$, their signs are to be compared on the intervals $t < 2$, $2 < t < 3$, $3 < t < 4$, and $t > 4$:
On the interval $t < 2$, $v > 0$ and $a < 0$; the speed is decreasing.
On the interval $2 < t < 3$, $v < 0$ and $a < 0$; the speed is increasing.
On the interval $3 < t < 4$, $v < 0$ and $a > 0$; the speed is decreasing.
On the interval $t > 4$, $v > 0$ and $a > 0$; the speed is increasing.
(*d*) When $t = 0$, $s = 0$ and the body is at O. The initial motion is to the right ($v > 0$) for the first 2 seconds; when $t = 2$, the body is $s = f(2) = 20$ ft from O.

During the next 2 seconds, it moves to the left, after which it is $s = f(4) = 16$ ft from O.

It then moves to the right, and after 5 seconds of motion in all, it is $s = f(5) = 20$ ft from O. The total distance traveled is $20 + 4 + 4 = 28$ ft (see Fig. 15-1.)

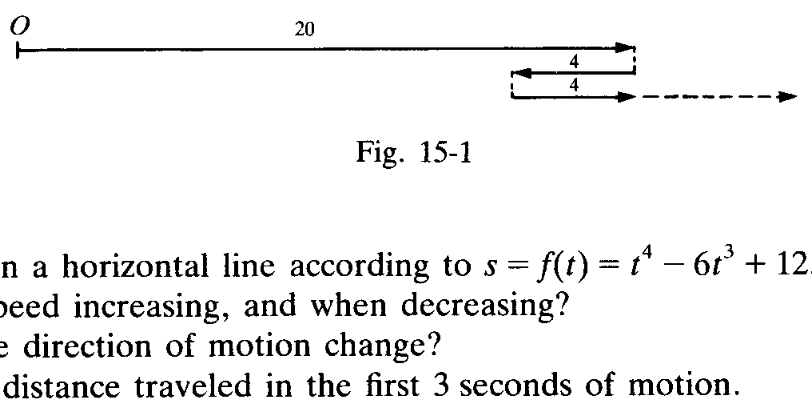


Fig. 15-1

4. A particle moves in a horizontal line according to $s = f(t) = t^4 - 6t^3 + 12t^2 - 10t + 3$.
(*a*) When is the speed increasing, and when decreasing?
(*b*) When does the direction of motion change?
(*c*) Find the total distance traveled in the first 3 seconds of motion.

Here

$$v = \frac{ds}{dt} = 4t^3 - 18t^2 + 24t - 10 = 2(t-1)^2(2t-5) \qquad a = \frac{dv}{dt} = 12(t-1)(t-2)$$

(*a*) v may change sign when $t = 1$ and $t = 2.5$; a may change sign when $t = 1$ and $t = 2$.
On the interval $t < 1$, $v < 0$ and $a > 0$; the speed is decreasing.
On the interval $1 < t < 2$, $v < 0$ and $a < 0$; the speed is increasing.
On the interval $2 < t < 2.5$, $v < 0$ and $a > 0$; the speed is decreasing.
On the interval $t > 2.5$, $v > 0$ and $a > 0$; the speed is increasing.
(*b*) The direction of motion changes at $t = 2.5$, since $v = 0$ but $a \neq 0$ there; it does not change at $t = 1$, since v does not change sign as t increases through $t = 1$. Note that when $t = 1$, $v = 0$ and $a = 0$, so that no information is available.

(*c*) When $t=0$, $s=3$ and the particle is 3 ft to the right of O. The motion is to the left for the first 2.5 seconds, after which the particle is $\frac{27}{16}$ ft to the left of O.

When $t=3$, $s=0$; the particle has moved $\frac{27}{16}$ ft to the right. The total distance traveled is $3+\frac{27}{16}+\frac{27}{16}=\frac{51}{8}$ ft (see Fig. 15-2).

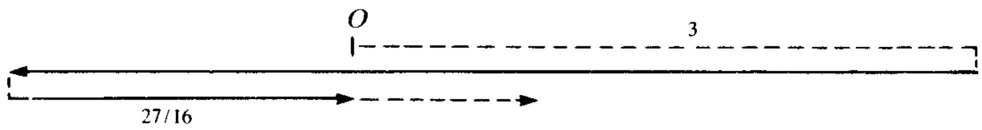


Fig. 15-2

5. A stone, projected vertically upward with initial velocity 112 ft/sec, moves according to $s=112t-16t^2$, where s is the distance from the starting point. Compute (*a*) the velocity and acceleration when $t=3$ and when $t=4$, and (*b*) the greatest height reached. (*c*) When will its height be 96 ft?

We have $v=ds/dt=112-32t$ and $a=dv/dt=-32$.

(*a*) At $t=3$, $v=16$ and $a=-32$. The stone is rising at 16 ft/sec.
At $t=4$, $v=-16$ and $a=-32$. The stone is falling at 16 ft/sec.

(*b*) At the highest point of the motion, $v=0$. Solving $v=0=112-32t$ yields $t=3.5$. At this time, $s=196$ ft.

(*c*) Letting $96=112t-16t^2$ yields $t^2-7t+6=0$, from which $t=1$ and 6. At the end of 1 second of motion the stone is at a height of 96 ft and is rising, since $v>0$. At the end of 6 seconds it is at the same height but is falling since $v<0$.

6. A particle rotates counterclockwise from rest according to $\theta=t^3/50-t$, where θ is in radians and t in seconds. Calculate the angular displacement θ, the angular velocity ω, and the angular acceleration α at the end of 10 seconds.

$$\theta=\frac{t^3}{50}-t=10\text{ rad}\qquad \omega=\frac{d\theta}{dt}=\frac{3t^2}{50}-1=5\text{ rad/sec}\qquad \alpha=\frac{d\omega}{dt}=\frac{6t}{50}=\frac{6}{5}\text{ rad/sec}^2$$

Supplementary Problems

7. A particle moves in a straight line according to $s=t^3-6t^2+9t$, the units being feet and seconds. Locate the particle with respect to its initial position ($t=0$) at O, find its direction and velocity, and determine whether its speed is increasing or decreasing when (*a*) $t=\frac{1}{2}$, (*b*) $t=\frac{3}{2}$, (*c*) $t=\frac{5}{2}$, (*d*) $t=4$.

Ans. (*a*) $\frac{25}{8}$ ft to the right of O; moving to the right with $v=\frac{15}{4}$ ft/sec; decreasing
(*b*) $\frac{27}{8}$ ft to the right of O; moving to the left with $v=-\frac{9}{4}$ ft/sec; increasing
(*c*) $\frac{5}{8}$ ft to the right of O; moving to the left with $v=-\frac{9}{4}$ ft/sec; decreasing
(*d*) 4 ft to the right of O; moving to the right with $v=9$ ft/sec; increasing

8. The distance of a locomotive from a fixed point on a straight track at time t is given by $s=3t^4-44t^3+144t^2$. When is it in reverse? *Ans.* $3<t<8$

9. Examine, as in Problem 2, each of the following straight-line motions: (*a*) $s=t^3-9t^2+24t$; (*b*) $s=t^3-3t^2+3t+3$; (*c*) $s=2t^3-12t^2+18t-5$; (*d*) $s=3t^4-28t^3+90t^2-108t$.

Ans. (*a*) stops at $t=2$ and $t=4$ with change of direction
(*b*) stops at $t=1$ without change of direction
(*c*) stops at $t=1$ and $t=3$ with change of direction
(*d*) stops at $t=1$ with, and $t=3$ without, change of direction

10. A body moves vertically up from the earth according to $s = 64t - 16t^2$. Show that it has lost one-half its velocity in its first 48 ft of rise.

11. A ball is thrown vertically upward from the edge of a roof in such a manner that it eventually falls to the street 112 ft below. If it moves so that its distance s from the roof at time t is given by $s = 96t - 16t^2$, find (*a*) the position of the ball, its velocity, and the direction of motion when $t = 2$, and (*b*) its velocity when it strikes the street. (s is in feet, and t in seconds.)

Ans. (*a*) 240 ft above the street, 32 ft/sec upward; (*b*) -128 ft/sec

12. A wheel turns through an angle θ radians in time t seconds so that $\theta = 128t - 12t^2$. Find the angular velocity and acceleration at the end of 3 sec. *Ans.* $\omega = 56$ rad/sec; $\alpha = -24$ rad/sec^2

13. Examine Problems 2 and 9 to conclude that stops with reversal of direction occur at values of t for which $s = f(t)$ has a maximum or minimum value while stops without reversal of direction occur at inflection points.

Chapter 16

Related Rates

RELATED RATES. If a quantity x is a function of time t, the *time rate of change* of x is given by dx/dt.

When two or more quantities, all functions of t, are related by an equation, the relation between their rates of change may be obtained by differentiating both sides of the equation.

Solved Problems

1. Gas is escaping from a spherical balloon at the rate of 2 ft^3/min. How fast is the surface area shrinking when the radius is 12 ft?

At time t the sphere has radius r, volume $V = \frac{4}{3}\pi r^3$, and surface $S = 4\pi r^2$. Then

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}. \quad \text{So} \quad \frac{dS}{dt} = \frac{2}{r}\frac{dV}{dt} = \frac{2}{12}(-2) = -\frac{1}{3}\ \text{ft}^2/\text{min}$$

2. Water is running out of a conical funnel at the rate of 1 in^3/sec. If the radius of the base of the funnel is 4 in and the altitude is 8 in, find the rate at which the water level is dropping when it is 2 in from the top.

Let r be the radius and h the height of the surface of the water at time t, and V the volume of water in the cone (see Fig. 16-1). By similar triangles, $r/4 = h/8$ or $r = \frac{1}{2}h$. Also

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3. \quad \text{So} \quad \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

When $dV/dt = -1$ and $h = 8 - 2 = 6$, then $dh/dt = -1/9\pi$ in/sec.

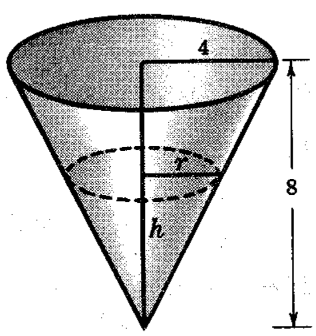


Fig. 16-1

3. Sand falling from a chute forms a conical pile whose altitude is always equal to $\frac{4}{3}$ the radius of the base. (*a*) How fast is the volume increasing when the radius of the base is 3 ft and is increasing at the rate of 3 in/min? (*b*) How fast is the radius increasing when it is 6 ft and the volume is increasing at the rate of 24 ft^3/min?

Let r be the radius of the base, and h the height of the pile at time t. Then

$$h = \frac{4}{3}r \quad \text{and} \quad V = \frac{1}{3}\pi r^2 h = \frac{4}{9}\pi r^3. \quad \text{So} \quad \frac{dV}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt}$$

(*a*) When $r = 3$ and $dr/dt = \frac{1}{4}$, $dV/dt = 3\pi$ ft^3/min.
(*b*) When $r = 6$ and $dV/dt = 24$, $dr/dt = 1/2\pi$ ft/min.

4. Ship A is sailing due south at 16 mi/h, and ship B, 32 miles south of A, is sailing due east at 12 mi/h. (*a*) At what rate are they approaching or separating at the end of 1 h? (*b*) At the end of 2 h? (*c*) When do they cease to approach each other, and how far apart are they at that time?

Let A_0 and B_0 be the initial positions of the ships, and A_t and B_t their positions t hours later. Let D be the distance between them t hours later. Then (see Fig. 16-2)

$$D^2 = (32 - 16t)^2 + (12t)^2 \qquad \text{and} \qquad \frac{dD}{dt} = \frac{400t - 512}{D}$$

(*a*) When $t = 1$, $D = 20$ and $dD/dt = -5.6$. They are approaching at 5.6 mi/h.
(*b*) When $t = 2$, $D = 24$ and $dD/dt = 12$. They are separating at 12 mi/h.
(*c*) They cease to approach each other when $dD/dt = 0$, that is, when $t = 512/400 = 1.28$ h, at which time they are $D = 19.2$ mi apart.

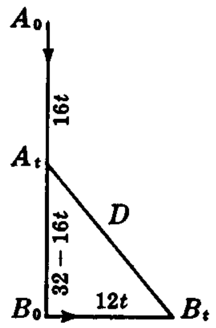


Fig. 16-2

5. Two parallel sides of a rectangle are being lengthened at the rate of 2 in/sec, while the other two sides are shortened in such a way that the figure remains a rectangle with constant area A = 50 in^2. What is the rate of change of the perimeter P when the length of an increasing side is (*a*) 5 in? (*b*) 10 in? (*c*) What are the dimensions when the perimeter ceases to decrease?

Let x be the length of the sides that are being lengthened, and y the length of the other sides, at time t. Then

$$P = 2(x + y) \qquad \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \qquad A = xy = 50 \qquad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

(*a*) When $x = 5$, $y = 10$ and $dx/dt = 2$. Then

$$5\frac{dy}{dt} + 10(2) = 0. \qquad \text{So} \qquad \frac{dy}{dt} = -4 \qquad \text{and} \qquad \frac{dP}{dt} = 2(2 - 4) = -4 \text{ in/sec (decreasing)}$$

(*b*) When $x = 10$, $y = 5$ and $dx/dt = 2$. Then

$$10\frac{dy}{dt} + 5(2) = 0. \qquad \text{So} \qquad \frac{dy}{dt} = -1 \qquad \text{and} \qquad \frac{dP}{dt} = 2(2 - 1) = 2 \text{ in/sec (increasing)}$$

(*c*) The perimeter will cease to decrease when $dP/dt = 0$, that is, when $dy/dt = -dx/dt = -2$. Then $x(-2) + y(2) = 0$, and the rectangle is a square of side $x = y = 5\sqrt{2}$ in.

6. The radius of a sphere is r in time t sec. Find the radius when the rates of increase of the surface area and the radius are numerically equal.

The surface area of the sphere is $S = 4\pi r^2$ so $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$. When $\frac{dS}{dt} = \frac{dr}{dt}$, $8\pi r = 1$ and the radius is $r = 1/8\pi$ in.

7. A weight W is attached to a rope 50 ft long that passes over a pulley at point P, 20 ft above the ground. The other end of the rope is attached to a truck at a point A, 2 ft above the ground as shown in Fig. 16-3. If the truck moves off at the rate of 9 ft/sec, how fast is the weight rising when it is 6 ft above the ground?

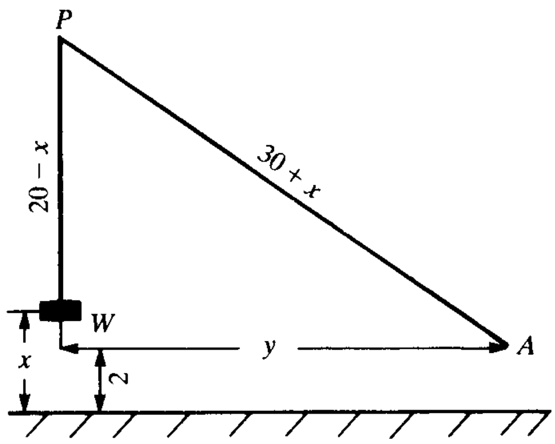


Fig. 16-3

Let x denote the distance the weight has been raised, and y the horizontal distance from point A, where the rope is attached to the truck, to the vertical line passing through the pulley. We must find dx/dt when $dy/dt = 9$ and $x = 6$.

Now

$$y^2 = (30 + x)^2 - (18)^2 \qquad \text{and} \qquad \frac{dy}{dt} = \frac{30 + x}{y}\frac{dx}{dt}$$

When $x = 6$, $y = 18\sqrt{3}$ and $dy/dt = 9$. Then $9 = \dfrac{30 + 6}{18\sqrt{3}}\dfrac{dx}{dt}$, from which $\dfrac{dx}{dt} = \dfrac{9}{2}\sqrt{3}$ ft/sec.

8. A light L hangs H ft above a street. An object h ft tall at O, directly under the light, is moved in a straight line along the street at v ft/sec. Investigate the velocity V of the tip of the shadow on the street after t sec. (See Fig. 16-4.)

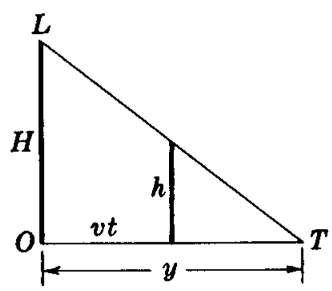


Fig. 16-4

After t seconds the object has been moved a distance vt. Let y be the distance of the tip of the shadow from O. Then

$$\frac{y - vt}{y} = \frac{h}{H} \qquad \text{or} \qquad y = \frac{Hvt}{H - h} \qquad \text{and so} \qquad V = \frac{dy}{dt} = \frac{Hv}{H - h} = \frac{1}{1 - h/H}\,v$$

Thus the velocity of the tip of the shadow is proportional to the velocity of the object, the factor of proportionality depending upon the ratio h/H. As $h \to 0$, $V \to v$, while as $h \to H$, V increases ever more rapidly.

Supplementary Problems

9. A rectangular trough is 8 ft long, 2 ft across the top, and 4 ft deep. If water flows in at a rate of 2 ft^3/min, how fast is the surface rising when the water is 1 ft deep? *Ans.* $\frac{1}{8}$ ft/min

10. A liquid is flowing into a vertical cylindrical tank of radius 6 ft at the rate of 8 ft^3/min. How fast is the surface rising? *Ans.* $2/9\pi$ ft/min

11. A man 5 ft tall walks at a rate of 4 ft/sec directly away from a street light that is 20 ft above the street. (*a*) At what rate is the tip of his shadow moving? (*b*) At what rate is the length of his shadow changing? *Ans.* (*a*) $\frac{16}{3}$ ft/sec; (*b*) $\frac{4}{3}$ ft/sec

12. A balloon is rising vertically over a point A on the ground at the rate of 15 ft/sec. A point B on the ground is level with and 30 ft from A. When the balloon is 40 ft from A, at what rate is its distance from B changing? *Ans.* 12 ft/sec

13. A ladder 20 ft long leans against a house. Find the rates at which (*a*) the top of the ladder is moving downward if its foot is 12 ft from the house and moving away at a rate of 2 ft/sec and (*b*) the slope of the ladder is decreasing. *Ans.* (*a*) $\frac{3}{2}$ ft/sec; (*b*) $\frac{25}{72}$ per sec

14. Water is being withdrawn from a conical reservoir 3 ft in radius and 10 ft deep at 4 ft^3/min. How fast is the surface falling when the depth of the water is 6 ft? How fast is the radius of this surface diminishing? *Ans.* $100/81\pi$ ft/min; $10/27\pi$ ft/min

15. A barge, whose deck is 10 ft below the level of a dock, is being drawn in by means of a cable attached to the deck and passing through a ring on the dock. When the barge is 24 ft away and approaching the dock at $\frac{3}{4}$ ft/sec, how fast is the cable being pulled in? (Neglect any sag in the cable.) *Ans.* $\frac{9}{13}$ ft/sec

16. A boy is flying a kite at a height of 150 ft. If the kite moves horizontally away from the boy at 20 ft/sec, how fast is the string being paid out when the kite is 250 ft from him? *Ans.* 16 ft/sec

17. One train, starting at 11 A.M., travels east at 45 mi/h while another, starting at noon from the same point, travels south at 60 mi/h. How fast are they separating at 3 P.M.? *Ans.* $105\sqrt{2}/2$ mi/h

18. A light is at the top of a pole 80 ft high. A ball is dropped at the same height from a point 20 ft from the light. Assuming that the ball falls according to $s = 16t^2$, how fast is the shadow of the ball moving along the ground 1 sec later? *Ans.* 200 ft/sec

19. Ship A is 15 mi east of O and moving west at 20 mi/h; ship B is 60 mi south of O and moving north at 15 mi/h. (*a*) Are they approaching or separating after 1 h and at what rate? (*b*) After 3 h? (*c*) When are they nearest one another?

Ans. (*a*) approaching, $115/\sqrt{82}$ mi/h; (*b*) separating, $9\sqrt{10}/2$ mi/h; (*c*) 1 h 55 min

20. Water, at a rate of 10 ft^3/min, is pouring into a leaky cistern whose shape is a cone 16 ft deep and 8 ft in diameter at the top. At the time the water is 12 ft deep, the water level is observed to be rising at 4 in/min. How fast is the water leaking away? *Ans.* $(10 - 3\pi)$ ft^3/min

21. A solution is passing through a conical filter 24 in deep and 16 in across the top, into a cylindrical vessel of diameter 12 in. At what rate is the level of the solution in the cylinder rising if, when the depth of the solution in the filter is 12 in, its level is falling at the rate 1 in/min? *Ans.* $\frac{4}{9}$ in/min

Chapter 17

Differentiation of Trigonometric Functions

RADIAN MEASURE. Let s denote the length of an arc AB intercepted by the central angle AOB on a circle of radius r, and let S denote the area of the sector AOB (see Fig. 17-1). (If s is $\frac{1}{360}$ of the circumference, then angle AOB has measure 1°; if $s = r$, angle AOB has measure 1 radian (rad). Recall that 1 rad $= 180/\pi$ degrees and $1° = \pi/180$ rad. Thus, $0° = 0$ rad; $30° = \pi/6$ rad; $45° = \pi/4$ rad; $180° = \pi$ rad; and $360° = 2\pi$ rad.)

Suppose $\angle AOB$ is measured as α degrees; then

$$s = \frac{\pi}{180}\alpha r \qquad \text{and} \qquad S = \frac{\pi}{360}\alpha r^2 \tag{17.1}$$

Suppose next that $\angle AOB$ is measured as θ radians; then

$$s = \theta r \qquad \text{and} \qquad S = \tfrac{1}{2}\theta r^2 \tag{17.2}$$

A comparison of (*17.1*) and (*17.2*) will make clear one of the advantages of radian measure.

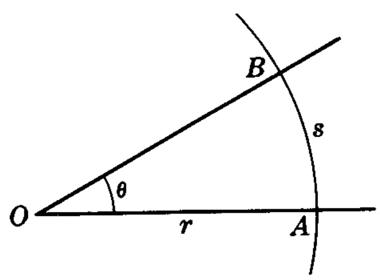


Fig. 17-1

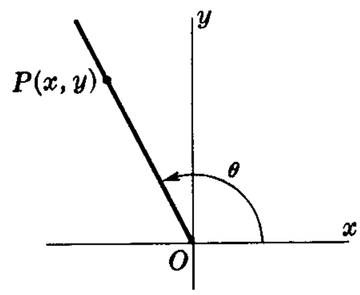


Fig. 17-2

TRIGONOMETRIC FUNCTIONS. Let θ be any real number. Construct the angle whose measure is θ radians with vertex at the origin of a rectangular coordinate system, and initial side along the positive x axis (Fig. 17-2). Take $P(x, y)$ on the terminal side of the angle a unit distance from O; then $\sin\theta = y$ and $\cos\theta = x$. The domain of definition of both $\sin\theta$ and $\cos\theta$ is the set of real numbers; the range of $\sin\theta$ is $-1 \le y \le 1$, and the range of $\cos\theta$ is $-1 \le x \le 1$. From

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{and} \qquad \sec\theta = \frac{1}{\cos\theta}$$

For both $\tan\theta$ and $\sec\theta$ the domain of definition ($\cos\theta \neq 0$) is $\theta \neq \pm\frac{2n-1}{2}\pi$, ($n = 1, 2, 3, \ldots$). It is left as an exercise for the reader to consider the functions

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \text{and} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Recall that, if θ is an acute angle of a right triangle ABC (Fig. 17-3), then

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{AB} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AC}{AB} \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{BC}{AC}$$

The slope m of a nonvertical line is equal to $\tan\alpha$, where α is the counterclockwise angle from the positive x axis to the line. (See Fig. 17-4.)

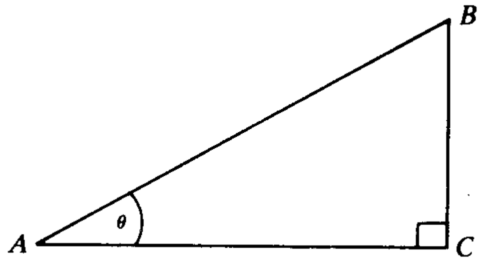


Fig. 17-3

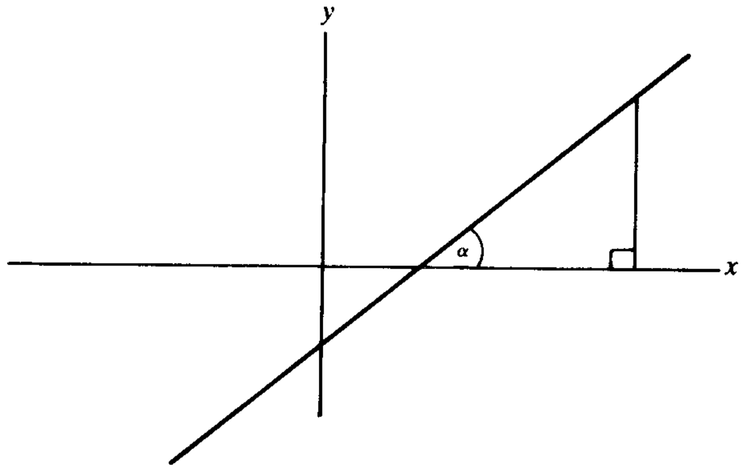


Fig. 17-4

Table 17-1 lists some standard trigonometric identities, and Table 17-2 contains some useful values of the trigonometric functions.

Table 17-1

$\sin^2\theta + \cos^2\theta = 1$

$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\sin(\alpha+2\pi) = \sin\alpha,\ \cos(\alpha+2\pi) = \cos\alpha$

$\sin(\alpha+\pi) = -\sin\alpha,\ \cos(\alpha+\pi) = -\cos\alpha,\ \tan(\alpha+\pi) = \tan\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha,\ \cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$

$\sin(\pi-\alpha) = \sin\alpha,\ \cos(\pi-\alpha) = -\cos\alpha$

$\sec^2\alpha = 1 + \tan^2\alpha$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Table 17-2

x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	∞
π	0	-1	0
$3\pi/2$	-1	0	∞

In Problem 1, we prove that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

(Had the angle been measured in degrees, the limit would have been $\pi/180$. This is another reason why radian measure is always used in the calculus.)

DIFFERENTIATION FORMULAS

14. $\dfrac{d}{dx}(\sin x) = \cos x$

15. $\dfrac{d}{dx}(\cos x) = -\sin x$

16. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

17. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

18. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

19. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

(See Problems 2 to 23.)

Solved Problems

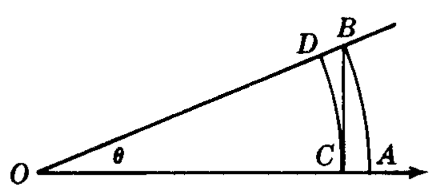
1. Prove: (*a*) $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ and (*b*) $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta} = 0$.

(*a*) Since $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$, we need consider only $\lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta}$. In Fig. 17-5, let $\theta = \angle AOB$ be a small positive central angle of a circle of radius $OA = 1$. Denote by C the foot of the perpendicular dropped from B onto OA, and by D the intersection of OB and an arc of radius OC. Now

$$\text{Sector } COD \le \triangle COB \le \text{sector } AOB$$

so that

$$\tfrac{1}{2}\theta\cos^2\theta \le \tfrac{1}{2}\sin\theta\cos\theta \le \tfrac{1}{2}\theta$$



$OC = \cos\theta$, $CB = \sin\theta$

Fig. 17-5

Dividing by $\frac{1}{2}\theta \cos\theta > 0$, we obtain

$$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}$$

Let $\theta \to 0^+$; then $\cos\theta \to 1$, $\dfrac{1}{\cos\theta} \to 1$, and $1 \le \lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} \le 1$; hence, $\lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1$.

(*b*)
$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}\,\frac{\cos\theta + 1}{\cos\theta + 1}$$
$$= \lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta\to 0} -\frac{\sin^2\theta}{\theta(\cos\theta + 1)}$$
$$= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \lim_{\theta\to 0} \frac{\sin\theta}{\cos\theta + 1} = -(1)\left(\frac{0}{2}\right) = 0$$

2. Derive: $\dfrac{d}{dx}(\sin x) = \cos x$.

Let $y = \sin x$. Then $y + \Delta y = \sin(x + \Delta x)$ and

$$\Delta y = \sin(x + \Delta x) - \sin x = \cos x \sin\Delta x + \sin x \cos\Delta x - \sin x$$
$$= \cos x \sin\Delta x + \sin x(\cos\Delta x - 1)$$
$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \left(\cos x \frac{\sin\Delta x}{\Delta x} + \sin x \frac{\cos\Delta x - 1}{\Delta x}\right)$$
$$= (\cos x) \lim_{\Delta x\to 0} \frac{\sin\Delta x}{\Delta x} + (\sin x) \lim_{\Delta x\to 0} \frac{\cos\Delta x - 1}{\Delta x}$$
$$= (\cos x)(1) + (\sin x)(0) = \cos x$$

3. Derive: $\dfrac{d}{dx}(\cos x) = -\sin x$.

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x$$

4. Derive: $\dfrac{d}{dx}(\tan x) = \sec^2 x$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

In Problems 5 to 12, find the first derivative.

5. $y = \sin 3x + \cos 2x$: $\quad y' = \cos 3x \dfrac{d}{dx}(3x) - \sin 2x \dfrac{d}{dx}(2x) = 3\cos 3x - 2\sin 2x$

6. $y = \tan x^2$: $\quad y' = \sec^2 x^2 \dfrac{d}{dx}(x^2) = 2x \sec^2 x^2$

7. $y = \tan^2 x = (\tan x)^2$: $\quad y' = 2\tan x \dfrac{d}{dx}(\tan x) = 2\tan x \sec^2 x$

8. $y = \cot(1 - 2x^2)$: $\quad y' = -\csc^2(1 - 2x^2) \dfrac{d}{dx}(1 - 2x^2) = 4x \csc^2(1 - 2x^2)$

9. $y = \sec^3\sqrt{x} = \sec^3 x^{1/2}$:

$$y' = 3\sec^2 x^{1/2}\frac{d}{dx}(\sec x^{1/2}) = 3\sec^2 x^{1/2}\sec x^{1/2}\tan x^{1/2}\frac{d}{dx}(x^{1/2}) = \frac{3}{2\sqrt{x}}\sec^3\sqrt{x}\tan\sqrt{x}$$

10. $\rho = \sqrt{\csc 2\theta} = (\csc 2\theta)^{1/2}$:

$$\rho' = \frac{1}{2}(\csc 2\theta)^{-1/2}\frac{d}{dx}(\csc 2\theta) = -\frac{1}{2}(\csc 2\theta)^{-1/2}(\csc 2\theta\cot 2\theta)(2) = -\sqrt{\csc 2\theta}\cot 2\theta$$

11. $f(x) = x^2\sin x$: $\quad f'(x) = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2) = x^2\cos x + 2x\sin x$

12. $f(x) = \dfrac{\cos x}{x}$: $\quad f'(x) = \dfrac{x\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(x)}{x^2} = \dfrac{-x\sin x - \cos x}{x^2}$

13. Let $y = x\sin x$; find y'''.

$$\begin{aligned} y' &= x\cos x + \sin x \\ y'' &= x(-\sin x) + \cos x + \cos x = -x\sin x + 2\cos x \\ y''' &= -x\cos x - \sin x - 2\sin x = -x\cos x - 3\sin x \end{aligned}$$

14. Let $y = \tan^2(3x-2)$; find y''.

$$\begin{aligned} y' &= 2\tan(3x-2)\sec^2(3x-2)\cdot 3 = 6\tan(3x-2)\sec^2(3x-2) \\ y'' &= 6[\tan(3x-2)\cdot 2\sec(3x-2)\cdot\sec(3x-2)\tan(3x-2)\cdot 3 + \sec^2(3x-2)\sec^2(3x-2)\cdot 3] \\ &= 36\tan^2(3x-2)\sec^2(3x-2) + 18\sec^4(3x-2) \end{aligned}$$

15. Let $y = \sin(x+y)$; find y'.

$$y' = \cos(x+y)\cdot(1+y'), \qquad \text{so that} \qquad y' = \frac{\cos(x+y)}{1-\cos(x+y)}$$

16. Let $\sin y + \cos x = 1$; find y''.

$$\cos y\cdot y' - \sin x = 0. \qquad \text{So} \qquad y' = \frac{\sin x}{\cos y}$$

Then
$$y'' = \frac{\cos y\cos x - \sin x(-\sin y)\cdot y'}{\cos^2 y} = \frac{\cos x\cos y + \sin x\sin y\cdot y'}{\cos^2 y}$$
$$= \frac{\cos x\cos y + \sin x\sin y(\sin x)/(\cos y)}{\cos^2 y} = \frac{\cos x\cos^2 y + \sin^2 x\sin y}{\cos^3 y}$$

17. Find $f'(\pi/3)$, $f''(\pi/3)$, and $f'''(\pi/3)$, given $f(x) = \sin x\cos 3x$.

$$\begin{aligned} f'(x) &= -3\sin x\sin 3x + \cos 3x\cos x \\ &= (\cos 3x\cos x - \sin 3x\sin x) - 2\sin x\sin 3x \\ &= \cos 4x - 2\sin x\sin 3x \end{aligned}$$

So $\quad f'(\pi/3) = -\frac{1}{2} - 2(\sqrt{3}/2)(0) = -\frac{1}{2}$

$$\begin{aligned} f''(x) &= -4\sin 4x - 2(3\sin x\cos 3x + \sin 3x\cos x) \\ &= -4\sin 4x - 2(\sin x\cos 3x + \sin 3x\cos x) - 4\sin x\cos 3x \\ &= -6\sin 4x - 4f(x) \end{aligned}$$

So $\quad f''(\pi/3) = -6(-\sqrt{3}/2) - 4(\sqrt{3}/2)(-1) = 5\sqrt{3}$

$f'''(x) = -24\cos 4x - 4f'(x)$. $\quad$ So $\quad f'''(\pi/3) = -24(-\frac{1}{2}) - 4(-\frac{1}{2}) = 14$

18. Find the acute angles of intersection of the curves (*1*) $y = 2\sin^2 x$ and (*2*) $y = \cos 2x$ on the interval $0 < x < 2\pi$. (See Fig. 17-6.)

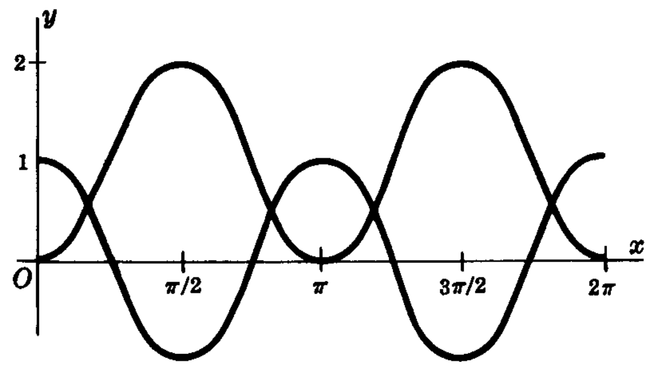


Fig. 17-6

We solve $2\sin^2 x = \cos 2x = 1 - 2\sin^2 x$ to obtain $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$ as the abscissas of the points of intersection.

Moreover, $y' = 4\sin x \cos x$ for (*1*), and $y' = -2\sin 2x$ for (*2*). Hence, at the point $\pi/6$, the curves have slopes $m_1 = \sqrt{3}$ and $m_2 = -\sqrt{3}$, respectively.

Since $\tan\phi = \dfrac{\sqrt{3}+\sqrt{3}}{1-3} = -\sqrt{3}$, the acute angle of intersection is 60°. At each of the remaining intersection points, the acute angle of intersection is also 60°.

19. A rectangular plot of ground has two adjacent sides along Highways 20 and 32. In the plot is a small lake, one end of which is 256 ft from Highway 20 and 108 ft from Highway 32 (see Fig. 17-7). Find the length of the shortest straight path which cuts across the plot from one highway to the other and touches the end of the lake.

Let s be the length of the path, and θ the angle it makes with Highway 32. Then

$$s = AP + PB = 108\csc\theta + 256\sec\theta$$

$$\frac{ds}{d\theta} = -108\csc\theta\cot\theta + 256\sec\theta\tan\theta = \frac{-108\cos^3\theta + 256\sin^3\theta}{\sin^2\theta\cos^2\theta}$$

Now $ds/d\theta = 0$ when $-108\cos^3\theta + 256\sin^3\theta = 0$, or when $\tan^3\theta = 27/64$, and the critical value is $\theta = \arctan 3/4$. Then $s = 108\csc\theta + 256\sec\theta = 108(5/3) + 256(5/4) = 500$ ft.

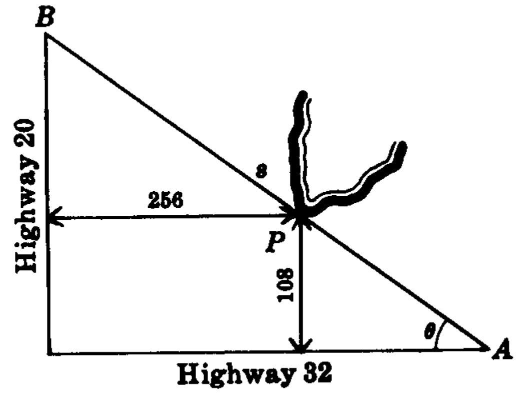


Fig. 17-7

20. Discuss the curve $y = f(x) = 4\sin x - 3\cos x$ on the interval $[0, 2\pi]$.

When $x = 0$, $y = f(0) = 4(0) - 3(1) = -3$.

Setting $f(x) = 0$ gives $\tan x = 3/4$, and the x-intercepts are $x = 0.64$ rad and $x = \pi + 0.64 = 3.78$ rad.

$f'(x) = 4\cos x + 3\sin x$. Setting $f'(x) = 0$ gives $\tan x = -\frac{4}{3}$, and the critical values are $x = \pi - 0.93 = 2.21$ and $x = 2\pi - 0.93 = 5.35$.

$f''(x) = -4\sin x + 3\cos x$. Setting $f''(x) = 0$ gives $\tan x = 3/4$, and the possible points of inflection are $x = 0.64$ and $x = \pi + 0.64 = 3.78$.

$f'''(x) = -4\cos x - 3\sin x$. In addition,

1. When $x = 2.21$, $\sin x = 4/5$ and $\cos x = -3/5$; then $f''(x) < 0$, so $x = 2.21$ yields a relative maximum of 5. $x = 5.35$ yields a relative minimum of -5.
2. $f'''(0.64) \neq 0$ and $f'''(3.78) \neq 0$. The points of inflection are $(0.64, 0)$ and $(3.78, 0)$.
3. The curve is concave upward from $x = 0$ to $x = 0.64$; concave downward from $x = 0.64$ to 3.78; and concave upward from $x = 3.78$ to 2π. (See Fig. 17-8.)

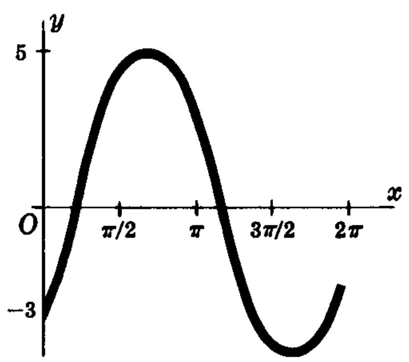


Fig. 17-8

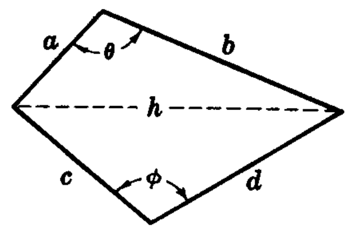


Fig. 17-9

21. Four bars of lengths a, b, c, and d are hinged together to form a quadrilateral (Fig. 17-9). Show that its area A is greatest when the opposite angles are supplementary.

Denote by θ the angle included by the bars of lengths a and b, by ϕ the opposite angle, and by h the length of the diagonal opposite these angles. We are required to maximize

$$A = \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\phi$$

subject to
$$h^2 = a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos\phi$$

Differentiation with respect to θ yields, respectively,

$$\frac{dA}{d\theta} = \frac{1}{2}ab\cos\theta + \frac{1}{2}cd\cos\phi\,\frac{d\phi}{d\theta} = 0 \qquad \text{and} \qquad ab\sin\theta = cd\sin\phi\,\frac{d\phi}{d\theta}$$

We solve for $d\phi/d\theta$ in the second of these equations and substitute in the first to obtain

$$ab\cos\theta + cd\cos\phi\,\frac{ab\sin\theta}{cd\sin\phi} = 0 \qquad \text{or} \qquad \sin\phi\cos\theta + \cos\phi\sin\theta = \sin(\phi + \theta) = 0$$

Then $\phi + \theta = 0$ or π, the first of which is easily rejected.

22. A bombardier is sighting on a target on the ground directly ahead. If the bomber is flying 2 mi above the ground at 240 mi/h, how fast must the sighting instrument be turning when the angle between the path of the bomber and the line of sight is 30°?

We have $dx/dt = -240$ mi/h, $\theta = 30°$, and $x = 2\cot\theta$ in Fig. 17-10. From the last equation,

$$\frac{dx}{dt} = -2\csc^2\theta\,\frac{d\theta}{dt} \qquad \text{or} \qquad -240 = -2(4)\frac{d\theta}{dt} \qquad \text{so} \qquad \frac{d\theta}{dt} = 30 \text{ rad/h} = \frac{3}{2\pi} \text{ degree/sec}$$

23. A ray of light passes through the air with velocity v_1 from a point P, a units above the surface of a body of water, to some point O on the surface and then with velocity v_2 to a point Q, b

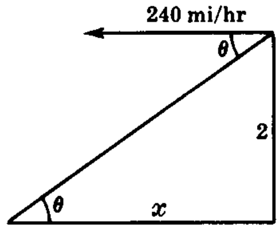


Fig. 17-10

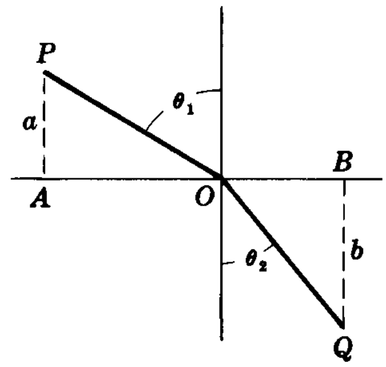


Fig. 17-11

units below the surface (Fig. 17-11). If OP and OQ make angles of θ_1 and θ_2 with a perpendicular to the surface, show that passage from P to Q is most rapid when $\sin\theta_1/\sin\theta_2 = v_1/v_2$.

Let t denote the time required for passage from P to Q, and c the distance from A to B; then

$$t = \frac{a\sec\theta_1}{v_1} + \frac{b\sec\theta_2}{v_2} \qquad \text{and} \qquad c = a\tan\theta_1 + b\tan\theta_2$$

Differentiating with respect to θ_1 yields

$$\frac{dt}{d\theta_1} = \frac{a\sec\theta_1\tan\theta_1}{v_1} + \frac{b\tan\theta_2\sec\theta_2}{v_2}\frac{d\theta_2}{d\theta_1} \qquad \text{and} \qquad 0 = a\sec^2\theta_1 + b\sec^2\theta_2\frac{d\theta_2}{d\theta_1}$$

From the last equation, $\dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\sec^2\theta_1}{b\sec^2\theta_2}$. For t to be a minimum, it is necessary that

$$\frac{dt}{d\theta_1} = \frac{a\sec\theta_1\tan\theta_1}{v_1} + \frac{b\sec\theta_2\tan\theta_2}{v_2}\left(-\frac{a\sec^2\theta_1}{b\sec^2\theta_2}\right) = 0$$

from which the required relation follows.

Supplementary Problems

24. Evaluate: (a) $\lim_{x\to 0}\dfrac{\sin 2x}{x} = 2\lim_{x\to 0}\dfrac{\sin 2x}{2x}$; ($b$) $\lim_{x\to 0}\dfrac{\sin ax}{\sin bx}$; ($c$) $\lim_{x\to 0}\dfrac{\sin^3 2x}{x\sin^2 3x}$.

Ans. (a) 2; (b) a/b; (c) 8/9

25. Derive differentiation formula 17, using first (a) $\cot u = \dfrac{\cos u}{\sin u}$ and then (b) $\cot u = \dfrac{1}{\tan u}$. Also derive differentiation formulas 18 and 19.

In Problems 26 to 45, find the derivative dy/dx or $d\rho/d\theta$.

26. $y = 3\sin 2x$ *Ans.* $6\cos 2x$

27. $y = 4\cos\frac{1}{2}x$ *Ans.* $-2\sin\frac{1}{2}x$

28. $y = 4\tan 5x$ *Ans.* $20\sec^2 5x$

29. $y = \frac{1}{4}\cot 8x$ *Ans.* $-2\csc^2 8x$

30. $y = 9\sec\frac{1}{3}x$ *Ans.* $3\sec\frac{1}{3}x\tan\frac{1}{3}x$

31. $y = \frac{1}{4}\csc 4x$ *Ans.* $-\csc 4x\cot 4x$

32. $y = \sin x - x\cos x + x^2 + 4x + 3$ *Ans.* $x\sin x + 2x + 4$

33. $\rho = \sqrt{\sin\theta}$ *Ans.* $(\cos\theta)/(2\sqrt{\sin\theta})$

34. $y = \sin 2/x$ *Ans.* $(-2\cos 2/x)/x^2$

35. $y = \cos(1 - x^2)$ *Ans.* $2x \sin(1 - x^2)$

36. $y = \cos(1 - x)^2$ *Ans.* $2(1 - x)\sin(1 - x)^2$

37. $y = \sin^2(3x - 2)$ *Ans.* $3\sin(6x - 4)$

38. $y = \sin^3(2x - 3)$ *Ans.* $-\frac{3}{2}\{\cos(6x - 9) - \cos(2x - 3)\}$

39. $y = \frac{1}{2}\tan x \sin 2x$ *Ans.* $\sin 2x$

40. $\rho = \dfrac{1}{(\sec 2\theta - 1)^{3/2}}$ *Ans.* $\dfrac{-3\sec 2\theta \tan 2\theta}{(\sec 2\theta - 1)^{5/2}}$

41. $\rho = \dfrac{\tan 2\theta}{1 - \cot 2\theta}$ *Ans.* $2\,\dfrac{\sec^2 2\theta - 4\csc 4\theta}{(1 - \cot 2\theta)^2}$

42. $y = x^2 \sin x + 2x\cos x - 2\sin x$ *Ans.* $x^2 \cos x$

43. $\sin y = \cos 2x$ *Ans.* $-2\sin 2x/\cos y$

44. $\cos 3y = \tan 2x$ *Ans.* $-2\sec^2 2x/3\sin 3y$

45. $x\cos y = \sin(x + y)$ *Ans.* $\dfrac{\cos y - \cos(x + y)}{x\sin y + \cos(x + y)}$

46. If $x = A\sin kt + B\cos kt$ for A, B, and k constants, show that $\dfrac{d^2x}{dt^2} = -k^2x$ and $\dfrac{d^{2n}x}{dt^{2n}} = (-1)^n k^{2n}x$.

47. Show: (*a*) $y'' + 4y = 0$ when $y = 3\sin(2x + 3)$; (*b*) $y''' + y'' + y' + y = 0$ when $y = \sin x + 2\cos x$.

48. Discuss and sketch on the interval $0 \le x < 2\pi$:
(*a*) $y = \frac{1}{2}\sin 2x$ (*b*) $y = \cos^2 x - \cos x$ (*c*) $y = x - 2\sin x$
(*d*) $y = \sin x(1 + \cos x)$ (*e*) $y = 4\cos^3 x - 3\cos x$

Ans. (*a*) maximum at $x = \pi/4, 5\pi/4$; minimum at $x = 3\pi/4, 7\pi/4$; inflection point at $x = 0, \pi/2, \pi, 3\pi/2$
(*b*) maximum at $x = 0, \pi$; minimum at $x = \pi/3, 5\pi/3$; inflection point at $x = 32°32', 126°23', 233°37', 327°28'$
(*c*) maximum at $x = 5\pi/3$; minimum at $x = \pi/3$; inflection point at $x = 0, \pi$
(*d*) maximum at $x = \pi/3$; minimum at $x = 5\pi/3$; inflection point at $x = 0, \pi, 104°29', 255°31'$
(*e*) maximum at $x = 0, 2\pi/3, 4\pi/3$; minimum at $x = \pi/3, \pi, 5\pi/3$; inflection point at $x = \pi/2, 3\pi/2, \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$

49. If the angle of elevation of the sun is 45° and is decreasing at $\frac{1}{4}$ rad/h, how fast is the shadow cast on level ground by a pole 50 ft tall lengthening? *Ans.* 25 ft/h

50. A kite, 120 ft above the ground, is moving horizontally at the rate of 10 ft/sec. At what rate is the inclination of the string to the horizontal diminishing when 240 ft of string are paid out?

Ans. $\frac{1}{48}$ rad/sec

51. A revolving beacon is situated 3600 ft off a straight shore. If the beacon turns at 4π rad/min, how fast does the beam sweep along the shore at (*a*) its nearest point, (*b*) at a point 4800 ft from the nearest point? *Ans.* (*a*) 240π ft/sec; (*b*) $2000\pi/3$ ft/sec

52. Two sides of a triangle are 15 and 20 ft long, respectively. (*a*) How fast is the third side increasing if the angle between the given sides is 60° and is increasing at the rate 2°/sec? (*b*) How fast is the area increasing? *Ans.* (*a*) $\pi/\sqrt{39}$ ft/sec; (*b*) $\frac{5}{6}\pi$ ft^2/sec

Chapter 18

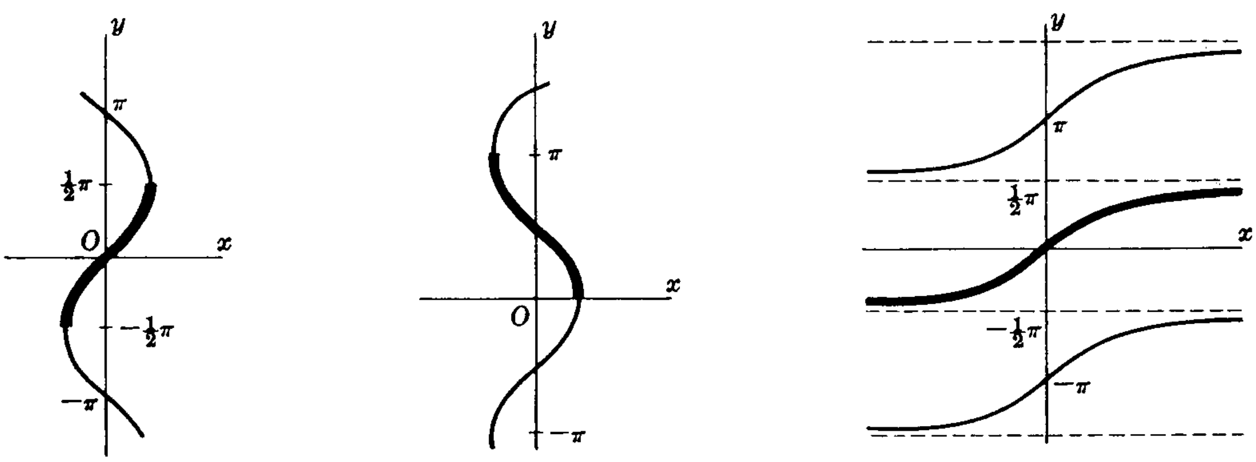
Differentiation of Inverse Trigonometric Functions

THE INVERSE TRIGONOMETRIC FUNCTIONS. If $x = \sin y$, the inverse function is written $y = \arcsin x$. (An alternative notation is $y = \sin^{-1} x$.) The domain of $\arcsin x$ is $-1 \le x \le 1$, which is the range of $\sin y$. The range of $\arcsin x$ is the set of real numbers, which is the domain of $\sin y$. The domain and range of the remaining inverse trigonometric functions may be established in a similar manner.

The inverse trigonometric functions are multivalued. In order that there be agreement on separating the graph into single-valued arcs, we define in Table 18-1 one such arc (called the *principal branch*) for each function. In Fig. 18-1, the principal branches are indicated by a thicker curve.

Table 18-1

Function	Principal Branch
$y = \arcsin x$	$-\frac{1}{2}\pi \leqq y \leqq \frac{1}{2}\pi$
$y = \arccos x$	$0 \leqq y \leqq \pi$
$y = \arctan x$	$-\frac{1}{2}\pi < y < \frac{1}{2}\pi$
$y = \operatorname{arccot} x$	$0 < y < \pi$
$y = \operatorname{arcsec} x$	$-\pi \leqq y < -\frac{1}{2}\pi,\ 0 \leqq y < \frac{1}{2}\pi$
$y = \operatorname{arccsc} x$	$-\pi < y \leqq -\frac{1}{2}\pi,\ 0 < y \leqq \frac{1}{2}\pi$



$y = \arcsin x$ $\quad$ $y = \arccos x$ $\quad$ $y = \arctan x$

Fig. 18-1

DIFFERENTIATION FORMULAS

20. $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$

21. $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$

22. $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$

23. $\dfrac{d}{dx}(\operatorname{arccot} x) = -\dfrac{1}{1+x^2}$

24. $\dfrac{d}{dx}(\operatorname{arcsec} x) = \dfrac{1}{x\sqrt{x^2-1}}$

25. $\dfrac{d}{dx}(\operatorname{arccsc} x) = -\dfrac{1}{x\sqrt{x^2-1}}$

Solved Problems

1. Derive: (a) $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$; (b) $\dfrac{d}{dx}(\operatorname{arcsec} x) = \dfrac{1}{x\sqrt{x^2-1}}$.

(a) Let $y = \arcsin x$. Then $x = \sin y$ and

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\frac{dy}{dx} = \cos y\,\frac{dy}{dx} = \sqrt{1-x^2}\,\frac{dy}{dx}$$

the sign being positive since $\cos y \ge 0$ on the interval $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$. Thus, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

(b) Let $y = \operatorname{arcsec} x$. Then $x = \sec y$ and

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}(\sec y) = \frac{d}{dy}(\sec y)\frac{dy}{dx} = \sec y \tan y\,\frac{dy}{dx} = x\sqrt{x^2-1}\,\frac{dy}{dx}$$

the sign being positive since $\tan y \ge 0$ on the intervals $0 \le y < \frac{1}{2}\pi$ and $-\pi \le y < -\frac{1}{2}\pi$. Thus, $\dfrac{d}{dx}(\operatorname{arcsec} x) = \dfrac{1}{x\sqrt{x^2-1}}$.

In Problems 2 to 8, find the first derivative.

2. $y = \arcsin(2x-3)$: $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(2x-3)^2}}\dfrac{d}{dx}(2x-3) = \dfrac{1}{\sqrt{3x-x^2-2}}$

3. $y = \arccos x^2$: $\quad \dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^4}}\dfrac{d}{dx}(x^2) = -\dfrac{2x}{\sqrt{1-x^4}}$

4. $y = \arctan 3x^2$: $\quad \dfrac{dy}{dx} = \dfrac{1}{1+(3x^2)^2}\dfrac{d}{dx}(3x^2) = \dfrac{6x}{1+9x^4}$

5. $f(x) = \operatorname{arccot}\dfrac{1+x}{1-x}$:

$$f'(x) = -\frac{1}{1+\left(\dfrac{1+x}{1-x}\right)^2}\frac{d}{dx}\left(\frac{1+x}{1-x}\right) = -\frac{1}{1+\left(\dfrac{1+x}{1-x}\right)^2}\,\frac{(1-x)-(1+x)(-1)}{(1-x)^2} = -\frac{1}{1+x^2}$$

6. $f(x) = x\sqrt{a^2-x^2} + a^2\arcsin\dfrac{x}{a}$:

$$f'(x) = x[\tfrac{1}{2}(a^2-x^2)^{-1/2}(-2x)] + (a^2-x^2)^{1/2} + a^2\frac{1}{\sqrt{1-(x/a)^2}}\,\frac{1}{a} = 2\sqrt{a^2-x^2}$$

7. $y = x\operatorname{arccsc}\dfrac{1}{x} + \sqrt{1-x^2}$:

$$y' = x\left[\frac{-1}{\dfrac{1}{x}\sqrt{\dfrac{1}{x^2}-1}}\,\frac{d}{dx}\left(\frac{1}{x}\right)\right] + \operatorname{arccsc}\frac{1}{x}\,\frac{d}{dx}(x) + \tfrac{1}{2}(1-x^2)^{-1/2}(-2x) = \operatorname{arccsc}\frac{1}{x}$$

8. $y = \dfrac{1}{ab}\arctan\left(\dfrac{b}{a}\tan x\right)$:

$$y' = \frac{1}{ab}\left[\frac{1}{1+\left(\frac{b}{a}\tan x\right)^2}\frac{d}{dx}\left(\frac{b}{a}\tan x\right)\right] = \frac{1}{ab}\,\frac{a^2}{a^2+b^2\tan^2 x}\,\frac{b}{a}\sec^2 x$$

$$= \frac{\sec^2 x}{a^2+b^2\tan^2 x} = \frac{1}{a^2\cos^2 x + b^2\sin^2 x}$$

9. $y^2 \sin x + y = \arctan x$; find y'.

$$2yy'\sin x + y^2\cos x + y' = \frac{1}{1+x^2}$$

Hence, $\quad y'(2y\sin x + 1) = \dfrac{1}{1+x^2} - y^2\cos x \quad$ and $\quad y' = \dfrac{1-(1+x^2)y^2\cos x}{(1+x^2)(2y\sin x+1)}$

10. In a circular arena (Fig. 18-2) there is a light at L. A boy starting from B runs at the rate of 10 ft/sec toward the center O. At what rate will his shadow be moving along the side when he is halfway from B to O?

Let P, a point x feet from B, be the position of the boy at time t; denote by r the radius of the arena, by θ the angle OLP, and by s the arc intercepted by θ. Then $s = r(2\theta)$, and $\theta = \arctan OP/LO = \arctan (r-x)/r$. Hence,

$$\frac{ds}{dt} = 2r\frac{d\theta}{dt} = 2r\frac{1}{1+[(r-x)/r]^2}\left(-\frac{1}{r}\right)\frac{dx}{dt} = \frac{-2r^2}{x^2-2rx+2r^2}\frac{dx}{dt}$$

When $x = \frac{1}{2}r$ and $dx/dt = 10$, $ds/dt = -16$ ft/sec. The shadow is moving along the wall at 16 ft/sec.

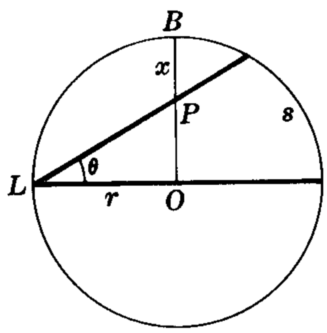


Fig. 18-2

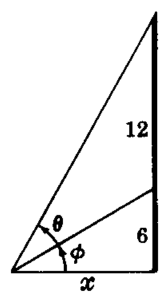


Fig. 18-3

11. The lower edge of a mural, 12 ft high, is 6 ft above an observer's eyes. Under the assumption that the most favorable view is obtained when the angle subtended by the mural at the eye is a maximum, at what distance from the wall should the observer stand?

Let θ denote the subtended angle, and x the distance from the wall. From Fig. 18-3, $\tan(\theta+\phi) = 18/x$, $\tan\phi = 6x$, and

$$\tan\theta = \tan[(\theta+\phi)-\phi] = \frac{\tan(\theta+\phi)-\tan\phi}{1+\tan(\theta+\phi)\tan\phi} = \frac{18/x - 6/x}{1+(18/x)(6/x)} = \frac{12x}{x^2+108}$$

Then $\theta = \arctan\dfrac{12x}{x^2+108}$ and $\dfrac{d\theta}{dx} = \dfrac{12(-x^2+108)}{x^4+360x^2+11{,}664}$. The critical value is $x = 6\sqrt{3} \sim 10.4$. The observer should stand 10.4 ft in front of the wall.

Supplementary Problems

12. Derive differentiation formulas 21, 22, 23, and 25.

In Problems 13 to 20, find dy/dx.

13. $y = \arcsin 3x$ *Ans.* $\dfrac{3}{\sqrt{1-9x^2}}$

14. $y = \arccos \frac{1}{2}x$ *Ans.* $-\dfrac{1}{\sqrt{4-x^2}}$

15. $y = \arctan \dfrac{3}{x}$ *Ans.* $-\dfrac{3}{x^2+9}$

16. $y = \arcsin (x-1)$ *Ans.* $\dfrac{1}{\sqrt{2x-x^2}}$

17. $y = x^2 \arccos 2/x$ *Ans.* $2x\left(\arccos \dfrac{2}{x} + \dfrac{1}{\sqrt{x^2-4}}\right)$

18. $y = \dfrac{x}{\sqrt{a^2-x^2}} - \arcsin \dfrac{x}{a}$ *Ans.* $\dfrac{x^2}{(a^2-x^2)^{3/2}}$

19. $y = (x-a)\sqrt{2ax-x^2} + a^2 \arcsin \dfrac{x-a}{a}$ *Ans.* $2\sqrt{2ax-x^2}$

20. $y = \dfrac{\sqrt{x^2-4}}{x^2} + \dfrac{1}{2} \operatorname{arcsec} \dfrac{x}{2}$ *Ans.* $\dfrac{8}{x^3\sqrt{x^2-4}}$

21. A light is to be placed directly above the center of a circular plot of radius 30 ft, at such a height that the edge of the plot will get maximum illumination. Find the height if the intensity at any point on the edge is directly proportional to the cosine of the angle of incidence (angle between the ray of light and the vertical) and inversely proportional to the square of the distance from the source. (*Hint*: Let x be the required height, y the distance from the light to a point on the edge, and θ the angle of incidence.) Then $I = k\dfrac{\cos\theta}{y^2} = \dfrac{kx}{(x^2+900)^{3/2}}$. *Ans.* $15\sqrt{2}$ ft

22. Two ships sail from A at the same time. One sails south at 15 mi/h; the other sails east at 25 mi/h for 1 h and then turns north. Find the rate of rotation of the line joining them after 3 h. *Ans.* $\frac{20}{193}$ rad/h

Chapter 19

Differentiation of Exponential and Logarithmic Functions

DEFINE THE NUMBER e by the equation

$$e = \lim_{h \to +\infty} \left(1 + \frac{1}{h}\right)^h$$

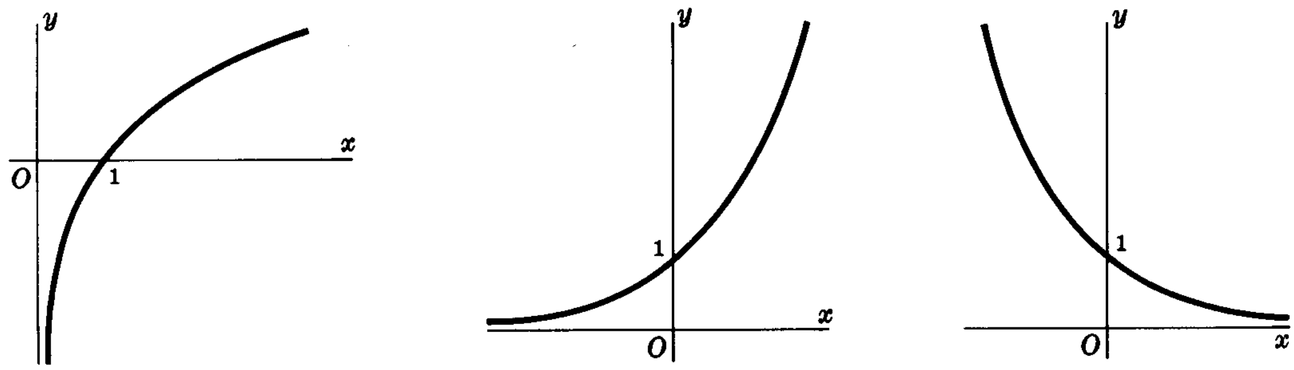
Then e also can be represented by $\lim_{k \to 0} (1 + k)^{1/k}$. In addition, it can be shown that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = 2.71828\ldots$$

The number e will serve as a base for the natural logarithm function (See Problem 1.)

LOGARITHMIC FUNCTIONS. Assume $a > 0$ and $a \neq 1$. If $a^y = x$, then define $y = \log_a x$. Another definition of $\log_a x$ will be given in Chapter 40.

NOTATION. Let $\ln x \equiv \log_e x$. (Then $\ln x$ is called the *natural logarithm* of x.) See also Fig. 19-1.
Let $\log x \equiv \log_{10} x$.
The domain of $\log_a x$ is $x > 0$; the range is the set of real numbers.



$y = \ln x$ $\qquad$ $y = e^{ax}$ $\qquad$ $y = e^{-ax}$

Fig. 19-1

DIFFERENTIATION FORMULAS

26. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x}\log_a e,\ a > 0,\ a \neq 1$ $\qquad$ 27. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

28. $\dfrac{d}{dx}(a^x) = a^x \ln a,\ a > 0$ $\qquad$ 29. $\dfrac{d}{dx}(e^x) = e^x$

(See Problems 2 to 17.)

LOGARITHMIC DIFFERENTIATION. If a differentiable function $y = f(x)$ is the product and/or quotient of several factors, the process of differentiation may be simplified by taking the natural

logarithm of the function before differentiation. This amounts to using the formula

30. $\dfrac{d}{dx}(y) = y\,\dfrac{d}{dx}(\ln y)$

(See Problems 18 and 19.)

BASIC PROPERTIES OF LOGARITHMS

Property 19.1: $\log_a 1 = 0$ (In particular, $\ln 1 = 0$.)

Property 19.2: $\log_a a = 1$ (In particular, $\ln e = 1$.)

Property 19.3: $\log_a uv = \log_a u + \log_a v$

Property 19.4: $\log_a \dfrac{u}{v} = \log_a u - \log_a v$

Property 19.5: $\log_a u^r = r \log_a u$

Solved Problems

1. Verify: $2 < \lim_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n < 3$.

By the binomial theorem, for n a positive integer,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\,\frac{1}{n} + \frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)(n-2)\cdots 1}{1\cdot 2\cdot 3\cdots n}\left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \left(1-\frac{1}{n}\right)\frac{1}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!} + \cdots + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\frac{1}{n!} \qquad (1)$$

Clearly, for every value of $n \neq 1$, $\left(1+\dfrac{1}{n}\right)^n > 2$. Also, if in (1) each difference $\left(1-\dfrac{1}{n}\right)$, $\left(1-\dfrac{2}{n}\right), \ldots$ is replaced by the larger number 1, we have

$$\left(1+\frac{1}{n}\right)^n < 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \quad \left(\text{Since } \frac{1}{n!} < \frac{1}{2^{n-1}}\right)$$

$$< 3 \quad \left(\text{Since } \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} < 1\right)$$

Hence, $2 < \left(1+\dfrac{1}{n}\right)^n < 3$.

Let $n \to \infty$ through positive integer values; then

$$1-\frac{1}{n}\to 1,\ 1-\frac{2}{n}\to 1,\ldots, \qquad \text{and} \qquad \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k}{n}\right)\frac{1}{k!}\to\frac{1}{k!}$$

This suggests that $\lim_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{k!} + \cdots = 2.71828\ldots$.

2. Derive $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x}\log_a e$ and $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

Let $y = \log_a x$. Then

$$y + \Delta y = \log_a (x + \Delta x)$$

$$\Delta y = \log_a (x + \Delta x) - \log_a x = \log_a \frac{x + \Delta x}{x} = \log_a \left(1 + \frac{\Delta x}{x}\right)$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) = \frac{1}{x} \frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) = \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

and
$$\frac{dy}{dx} = \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} = \frac{1}{x} \log_a \left[\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right] = \frac{1}{x} \log_a e$$

When $a = e$, $\log_a e = \log_e e = 1$ and $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

In Problems 3 to 9, find the first derivative.

3. $y = \log_a (3x^2 - 5)$:
$$\frac{dy}{dx} = \frac{1}{3x^2 - 5} (\log_a e) \frac{d}{dx}(3x^2 - 5) = \frac{6x}{3x^2 - 5} \log_a e$$

4. $y = \ln (x + 3)^2 = 2 \ln (x + 3)$:
$$\frac{dy}{dx} = 2 \frac{1}{x + 3} \frac{d}{dx}(x + 3) = \frac{2}{x + 3}$$

5. $y = \ln^2 (x + 3)$:
$$y' = 2 \ln (x + 3) \frac{d}{dx}[\ln (x + 3)] = 2 \ln (x + 3) \frac{1}{x + 3} \frac{d}{dx}(x + 3) = \frac{2 \ln (x + 3)}{x + 3}$$

6. $y = \ln (x^3 + 2)(x^2 + 3) = \ln (x^3 + 2) + \ln (x^2 + 3)$:
$$y' = \frac{1}{x^3 + 2} \frac{d}{dx}(x^3 + 2) + \frac{1}{x^2 + 3} \frac{d}{dx}(x^2 + 3) = \frac{3x^2}{x^3 + 2} + \frac{2x}{x^2 + 3}$$

7. $f(x) = \ln \dfrac{x^4}{(3x - 4)^2} = \ln x^4 - \ln (3x - 4)^2 = 4 \ln x - 2 \ln (3x - 4)$:
$$f'(x) = 4 \frac{1}{x} \frac{d}{dx}(x) - 2 \frac{1}{3x - 4} \frac{d}{dx}(3x - 4) = \frac{4}{x} - \frac{6}{3x - 4}$$

8. $y = \ln \sin 3x$:
$$y' = \frac{1}{\sin 3x} \frac{d}{dx}(\sin 3x) = 3 \frac{\cos 3x}{\sin 3x} = 3 \cot 3x$$

9. $y = \ln (x + \sqrt{1 + x^2})$:
$$y' = \frac{1 + \frac{1}{2}(1 + x^2)^{-1/2}(2x)}{x + (1 + x^2)^{1/2}} = \frac{1 + x(1 + x^2)^{-1/2}}{x + (1 + x^2)^{1/2}} \frac{(1 + x^2)^{1/2}}{(1 + x^2)^{1/2}} = \frac{1}{\sqrt{1 + x^2}}$$

10. Derive $\dfrac{d}{dx}(a^x) = (\ln a)a^x$ and $\dfrac{d}{dx}(e^x) = e^x$.

Let $y = a^x$. Then $\ln y = x \ln a$ and
$$\frac{d}{dx}(\ln y) = \frac{1}{y} \frac{dy}{dx} = \ln a \qquad \text{or} \qquad \frac{dy}{dx} = y \ln a = a^x \ln a$$

When $a = e$, $\ln a = \ln e = 1$ and we have $\dfrac{d}{dx}(e^x) = e^x$.

In Problems 11 to 15, find the first derivative.

11. $y = e^{-\frac{1}{2}x}$: $\qquad y' = e^{-\frac{1}{2}x} \dfrac{d}{dx}\left(-\dfrac{1}{2}x\right) = -\dfrac{1}{2}e^{-\frac{1}{2}x}$

12. $y = e^{x^2}$: $\qquad y' = e^{x^2} \dfrac{d}{dx}(x^2) = 2xe^{x^2}$

13. $y = a^{3x^2}$: $\qquad y' = a^{3x^2}(\ln a) \dfrac{d}{dx}(3x^2) = 6xa^{3x^2} \ln a$

14. $y = x^2 3^x$: $\qquad y' = x^2 \dfrac{d}{dx}(3^x) + 3^x \dfrac{d}{dx}(x^2) = x^2 3^x \ln 3 + 3^x 2x = x3^x(x \ln 3 + 2)$

15. $y = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$:

$$y' = \frac{(e^{ax} + e^{-ax})\dfrac{d}{dx}(e^{ax} - e^{-ax}) - (e^{ax} - e^{-ax})\dfrac{d}{dx}(e^{ax} + e^{-ax})}{(e^{ax} + e^{-ax})^2}$$
$$= \frac{(e^{ax} + e^{-ax})(a)(e^{ax} + e^{-ax}) - (e^{ax} - e^{-ax})(a)(e^{ax} - e^{-ax})}{(e^{ax} + e^{-ax})^2}$$
$$= a\,\frac{(e^{2ax} + 2 + e^{-2ax}) - (e^{2ax} - 2 + e^{-2ax})}{(e^{ax} + e^{-ax})^2} = \frac{4a}{(e^{ax} + e^{-ax})^2}$$

16. Find y'', given $y = e^{-x} \ln x$.

$$y' = e^{-x} \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(e^{-x}) = \frac{e^{-x}}{x} - e^{-x} \ln x = \frac{e^{-x}}{x} - y$$
$$y'' = \frac{x \dfrac{d}{dx}(e^{-x}) - e^{-x} \dfrac{d}{dx}(x)}{x^2} - y' = \frac{-xe^{-x} - e^{-x}}{x^2} - \frac{e^{-x}}{x} + e^{-x} \ln x = -e^{-x}\left(\frac{2}{x} + \frac{1}{x^2} - \ln x\right)$$

17. Find y'', given $y = e^{-2x} \sin 3x$.

$$y' = e^{-2x} \frac{d}{dx}(\sin 3x) + \sin 3x \frac{d}{dx}(e^{-2x}) = 3e^{-2x} \cos 3x - 2e^{-2x} \sin 3x = 3e^{-2x} \cos 3x - 2y$$
$$y'' = 3e^{-2x} \frac{d}{dx}(\cos 3x) + 3 \cos 3x \frac{d}{dx}(e^{-2x}) - 2y'$$
$$= -9e^{-2x} \sin 3x - 6e^{-2x} \cos 3x - 2(3e^{-2x} \cos 3x - 2e^{-2x} \sin 3x)$$
$$= -e^{-2x}(12 \cos 3x + 5 \sin 3x)$$

In Problems 18 and 19, use logarithmic differentiation to find the first derivative.

18. $y = (x^2 + 2)^3(1 - x^3)^4$

$$\ln y = \ln (x^2 + 2)^3(1 - x^3)^4 = 3 \ln (x^2 + 2) + 4 \ln (1 - x^3)$$
$$y' = y \frac{d}{dx}[3 \ln (x^2 + 2) + 4 \ln (1 - x^3)] = (x^2 + 2)^3(1 - x^3)^4\left(\frac{6x}{x^2 + 2} - \frac{12x^2}{1 - x^3}\right)$$
$$= 6x(x^2 + 2)^2(1 - x^3)^3(1 - 4x - 3x^3)$$

19. $y = \dfrac{x(1 - x^2)^2}{(1 + x^2)^{1/2}}$

$$\ln y = \ln x + 2\ln(1-x^2) - \tfrac{1}{2}\ln(1+x^2)$$

$$y' = \frac{x(1-x^2)^2}{(1+x^2)^{1/2}}\left(\frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}\right) = \frac{(1-x^2)^2}{(1+x^2)^{1/2}} - \frac{4x^2(1-x^2)}{(1+x^2)^{1/2}} - \frac{x^2(1-x^2)^2}{(1+x^2)^{3/2}}$$

$$= \frac{(1-5x^2-4x^4)(1-x^2)}{(1+x^2)^{3/2}}$$

20. Locate (*a*) the relative maximum and minimum points and (*b*) the points of inflection of the curve $y = f(x) = x^2e^x$ (Fig. 19-2).

$$f'(x) = 2xe^x + x^2e^x = xe^x(2+x)$$
$$f''(x) = 2e^x + 4xe^x + x^2e^x = e^x(2+4x+x^2)$$
$$f'''(x) = 6e^x + 6xe^x + x^2e^x = e^x(6+6x+x^2)$$

(*a*) Solving $f'(x) = 0$ gives the critical values $x = 0$ and $x = -2$. Then $f''(0) > 0$; so $(0, 0)$ is a relative minimum point. Also, $f''(-2) < 0$; so $(-2, 4/e^2)$ is a relative maximum point.

(*b*) Solving $f''(x) = 0$ gives possible points of inflection at $x = -2 \pm \sqrt{2}$. Since $f'''(-2-\sqrt{2}) \neq 0$ and $f'''(-2+\sqrt{2}) \neq 0$, the points at $x = -2 \pm \sqrt{2}$ are points of inflection.

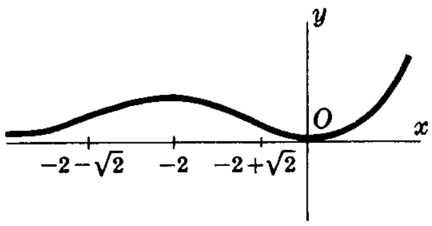


Fig. 19-2

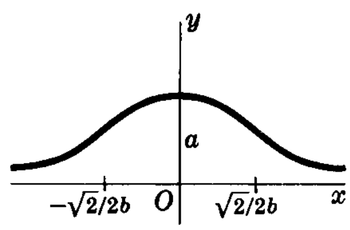


Fig. 19-3

21. Discuss the probability curve $y = ae^{-b^2x^2}$, $a > 0$ (Fig. 19-3).

The curve lies entirely above the x axis, since $e^{-b^2x^2} > 0$ for all x. As $x \to \pm\infty$, $y \to 0$; hence the x axis is a horizontal asymptote.

The first two derivatives are

$$y' = -2ab^2xe^{-b^2x^2} \quad \text{and} \quad y'' = 2ab^2(2b^2x^2 - 1)e^{-b^2x^2}$$

When $y' = 0$, $x = 0$, and when $x = 0$, $y'' < 0$. Hence the point $(0, a)$ is a maximum point of the curve. When $y'' = 0$, $2b^2x^2 - 1 = 0$, yielding $x = \pm\sqrt{2}/2b$ as possible points of inflection. We have:

$-\sqrt{2}/2b$	0	$\sqrt{2}/2b$
$y'' > 0$ concave up	$y'' < 0$ concave down	$y'' > 0$ concave up

Hence the points $(\pm\sqrt{2}/2b, ae^{-1/2})$ are points of inflection.

22. The equilibrium constant K of a balanced chemical reaction changes with the absolute temperature T according to $K = K_0e^{-\frac{1}{2}q(T-T_0)/T_0T}$, where K_0, q, and T_0 are constants. Find the percentage rate of change of K per degree of change of T.

The percentage rate of change of K per degree of change of T is given by $\dfrac{100}{K}\dfrac{dK}{dT} = 100\dfrac{d}{dt}(\ln K)$. Then,

$$\ln K = \ln K_0 - \frac{1}{2}q\frac{T-T_0}{T_0T} \quad \text{and} \quad 100\frac{d}{dT}(\ln K) = -\frac{100q}{2T^2} = -\frac{50q}{T^2}\,\%$$

23. Discuss the damped-vibration curve $y = f(t) = e^{-\frac{1}{2}t} \sin 2\pi t$.

When $t = 0$, $y = 0$. The y intercept is thus 0.

When $y = 0$, we have $\sin 2\pi t = 0$ and $t = \ldots, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. These are the t intercepts.

When $t = \ldots, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \ldots$, we have $\sin 2\pi t = 1$ and $y = e^{-\frac{1}{2}t}$. When $t = \ldots, -\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \ldots$, we have $\sin 2\pi t = -1$ and $y = -e^{-\frac{1}{2}t}$. The given curve oscillates between the two curves $y = e^{-\frac{1}{2}t}$ and $y = -e^{-\frac{1}{2}t}$, touching them at these points, as shown in Fig. 19-4.

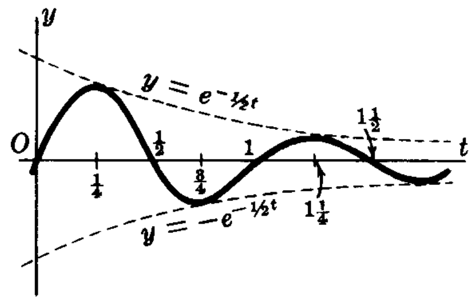


Fig. 19-4

Differentiation yields

$$y' = f'(t) = e^{-\frac{1}{2}t}(2\pi \cos 2\pi t - \tfrac{1}{2} \sin 2\pi t)$$
$$y'' = f''(t) = e^{-\frac{1}{2}t}[(\tfrac{1}{4} - 4\pi^2) \sin 2\pi t - 2\pi \cos 2\pi t]$$

When $y' = 0$, then $2\pi \cos 2\pi t - \frac{1}{2} \sin 2\pi t = 0$; that is, $\tan 2\pi t = 4\pi$. If $t = \xi = 0.237$ is the smallest positive angle satisfying this relation, then $t = \ldots, \xi - \frac{3}{2}, \xi - 1, \xi - \frac{1}{2}, \xi, \xi + \frac{1}{2}, \xi + 1, \ldots$ are the critical values.

For $n = 0, 1, 2, \ldots$, $f''(\xi \pm \frac{1}{2}n)$ and $f''\left(\xi \pm \dfrac{n+1}{2}\right)$ have opposite signs, whereas $f''(\xi \pm \frac{1}{2}n)$ and $f''\left(\xi \pm \dfrac{n+2}{2}\right)$ have the same sign; hence, the critical values yield alternate maximum and minimum points of the curve. These points are slightly to the left of the points of contact with the curves $y = e^{-\frac{1}{2}t}$ and $y = -e^{-\frac{1}{2}t}$.

When $y'' = 0$, $\tan 2\pi t = \dfrac{2\pi}{1/4 - 4\pi^2} = \dfrac{8\pi}{1 - 16\pi^2}$. If $t = \eta = 0.475$ is the smallest positive angle satisfying this relation, then $t = \ldots, \eta - 1, \eta - \frac{1}{2}, \eta, \eta + \frac{1}{2}, \eta + 1, \ldots$ are the possible points of inflection. These points, located slightly to the left of the points of intersection of the curve and the t axis, are points of inflection.

24. The equation $s = ce^{-bt} \sin(kt + \theta)$, where c, b, k, and θ are constants, represents damped vibratory motion. Show that $a = -2bv - (k^2 + b^2)s$, where $v = ds/dt$ and $a = dv/dt$.

$$v = \frac{ds}{dt} = ce^{-bt}[-b \sin(kt + \theta) + k \cos(kt + \theta)]$$

$$a = \frac{dv}{dt} = ce^{-bt}[(b^2 - k^2) \sin(kt + \theta) - 2bk \cos(kt + \theta)]$$
$$= ce^{-bt}\{-2b[-b \sin(kt + \theta) + k \cos(kt + \theta)] - (k^2 + b^2) \sin(kt + \theta)\}$$
$$= -2bv - (k^2 + b^2)s$$

Supplementary Problems

In Problems 25 to 35, find dy/dx.

25. $y = \ln(4x - 5)$ *Ans.* $4/(4x - 5)$

26. $y = \ln\sqrt{3 - x^2}$ *Ans.* $x/(x^2 - 3)$

27. $y = \ln 3x^5$ *Ans.* $5/x$

28. $y = \ln (x^2 + x - 1)^3$ *Ans.* $(6x + 3)/(x^2 + x - 1)$

29. $y = x \cdot \ln x - x$ *Ans.* $\ln x$

30. $y = \ln (\sec x + \tan x)$ *Ans.* $\sec x$

31. $y = \ln (\ln \tan x)$ *Ans.* $2/(\sin 2x \ln \tan x)$

32. $y = (\ln x^2)/x^2$ *Ans.* $(2 - 4 \ln x)/x^3$

33. $y = \frac{1}{5}x^5(\ln x - \frac{1}{5})$ *Ans.* $x^4 \ln x$

34. $y = x[\sin (\ln x) - \cos (\ln x)]$ *Ans.* $2 \sin (\ln x)$

35. $y = x \ln (4 + x^2) + 4 \arctan \frac{1}{2}x - 2x$ *Ans.* $\ln (4 + x^2)$

36. Find the equation of the line tangent to $y = \ln x$ at any one of its points (x_0, y_0). Use the y intercept of the tangent line to obtain a simple construction for the tangent line.

Ans. $y - y_0 = (1/x_0)(x - x_0)$

37. Discuss and sketch: $y = x^2 \ln x$. *Ans.* minimum at $x = 1/\sqrt{e}$; inflection point at $x = 1/e^{3/2}$

38. Show that the angle of intersection of the curves $y = \ln (x - 2)$ and $y = x^2 - 4x + 3$ at the point $(3, 0)$ is $\phi = \arctan \frac{1}{3}$.

In Problems 39 to 46, find dy/dx.

39. $y = e^{5x}$ *Ans.* $5e^{5x}$

40. $y = e^{x^3}$ *Ans.* $3x^2e^{x^3}$

41. $y = e^{\sin 3x}$ *Ans.* $3e^{\sin 3x} \cos 3x$

42. $y = 3^{-x^2}$ *Ans.* $-2x(3^{-x^2} \ln 3)$

43. $y = e^{-x} \cos x$ *Ans.* $-e^{-x}(\cos x + \sin x)$

44. $y = \arcsin e^x$ *Ans.* $e^x/\sqrt{1 - e^{2x}}$

45. $y = \tan^2 e^{3x}$ *Ans.* $6e^{3x} \tan e^{3x} \sec^2 e^{3x}$

46. $y = e^{e^x}$ *Ans.* $e^{(x + e^x)}$

47. If $y = x^2e^x$, show that $y''' = (x^2 + 6x + 6)e^x$.

48. If $y = e^{-2x}(\sin 2x + \cos 2x)$, show that $y'' + 4y' + 8y = 0$.

49. Discuss and sketch: (*a*) $y = x^2e^{-x}$ and (*b*) $y = x^2e^{-x^2}$.

Ans. (*a*) maximum at $x = 2$; minimum at $x = 0$; inflection points at $x = 2 \pm \sqrt{2}$
(*b*) maximum at $x = \pm 1$; minimum at $x = 0$; inflection points at $x = \pm 1.51$, $x = \pm 0.47$

50. Find the rectangle of maximum area, having one edge along the x axis, under the curve $y = e^{-x^2}$. (*Hint*: $A = 2xy = 2xe^{-x^2}$, where $P(x, y)$ is a vertex of the rectangle on the curve.) *Ans.* $A = \sqrt{2/e}$

51. Show that the curves $y = e^{ax}$ and $y = e^{ax} \cos ax$ are tangent at the points for which $x = 2n\pi/a$ $(n = 1, 2, 3, \ldots)$, and that the curves $y = e^{-ax}/a^2$ and $y = e^{ax} \cos ax$ are mutually perpendicular at the same points.

52. For the curve $y = xe^x$, show (*a*) $(-1, -1/e)$ is a relative minimum point, (*b*) $(-2, -2/e^2)$ is a point of inflection, and (*c*) the curve is concave downward to the left of the point of inflection, and concave upward to the right of it.

In Problems 53 to 56, use logarithmic differentiation to find dy/dx.

53. $y = x^x$ *Ans.* $x^x(1 + \ln x)$

54. $y = x^2e^{2x}\cos 3x$ *Ans.* $x^2e^{2x}\cos 3x(2/x + 2 - 3\tan 3x)$

55. $y = x^{\ln x}$ *Ans.* $2x^{(\ln x - 1)}\ln x$

56. $y = x^{e^{-x^2}}$ *Ans.* $e^{-x^2}x^{e^{-x^2}}(1/x - 2x\ln x)$

57. Show (*a*) $\dfrac{d^n}{dx^n}(xe^x) = (x + n)e^x$; (*b*) $\dfrac{d^n}{dx^n}(x^{n-1}\ln x) = \dfrac{(n-1)!}{x}$.

Chapter 20

Differentiation of Hyperbolic Functions

DEFINITIONS OF HYPERBOLIC FUNCTIONS. For x any real number, except where noted, the hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \neq 0$$

DIFFERENTIATION FORMULAS

31. $\dfrac{d}{dx}(\sinh x) = \cosh x$

32. $\dfrac{d}{dx}(\cosh x) = \sinh x$

33. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

34. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

35. $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

36. $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

(See Problems 1 to 12.)

DEFINITIONS OF INVERSE HYPERBOLIC FUNCTIONS

$$\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right) \quad \text{for all } x \qquad \coth^{-1} x = \tfrac{1}{2} \ln \frac{x+1}{x-1}, \quad x^2 > 1$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1 \qquad \operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x}, \quad 0 < x \leq 1$$

$$\tanh^{-1} x = \tfrac{1}{2} \ln \frac{1+x}{1-x}, \quad x^2 < 1 \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$$

(Only principal values of $\cosh^{-1} x$ and $\operatorname{sech}^{-1} x$ are included here.)

DIFFERENTIATION FORMULAS

37. $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$

38. $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}, \quad x > 1$

39. $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}, \quad x^2 < 1$

40. $\dfrac{d}{dx}(\coth^{-1} x) = \dfrac{1}{1-x^2}, \quad x^2 > 1$

41. $\dfrac{d}{dx}(\operatorname{sech}^{-1} x) = \dfrac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

42. $\dfrac{d}{dx}(\operatorname{csch}^{-1} x) = \dfrac{-1}{|x|\sqrt{1+x^2}}, \quad x \neq 0$

(See Problems 13 to 19.)

Solved Problems

1. Prove that $\cosh^2 u - \sinh^2 u = 1$.

$$\cosh^2 u - \sinh^2 u = \left(\frac{e^u + e^{-u}}{2}\right)^2 - \left(\frac{e^u - e^{-u}}{2}\right)^2 = \tfrac{1}{4}(e^{2u} + 2 + e^{-2u}) - \tfrac{1}{4}(e^{2u} - 2 + e^{-2u}) = 1$$

2. Derive $\frac{d}{dx}(\sinh x) = \cosh x$.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

In Problems 3 to 10, find dy/dx.

3. $y = \sinh 3x$: $\quad \frac{dy}{dx} = \cosh 3x \frac{d}{dx}(3x) = 3\cosh 3x$

4. $y = \cosh \frac{1}{2}x$: $\quad \frac{dy}{dx} = \sinh \frac{1}{2}x \frac{d}{dx}(\frac{1}{2}x) = \frac{1}{2}\sinh \frac{1}{2}x$

5. $y = \tanh(1 + x^2)$: $\quad \frac{dy}{dx} = \operatorname{sech}^2(1 + x^2)\frac{d}{dx}(1 + x^2) = 2x \operatorname{sech}^2(1 + x^2)$

6. $y = \coth \frac{1}{x}$: $\quad \frac{dy}{dx} = -\operatorname{csch}^2 \frac{1}{x} \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2}\operatorname{csch}^2 \frac{1}{x}$

7. $y = x \operatorname{sech} x^2$:

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}(\operatorname{sech} x^2) + \operatorname{sech} x^2 \frac{d}{dx}(x) = x(-\operatorname{sech} x^2 \tanh x^2)2x + \operatorname{sech} x^2 \\ &= -2x^2 \operatorname{sech} x^2 \tanh x^2 + \operatorname{sech} x^2\end{aligned}$$

8. $y = \operatorname{csch}^2(x^2 + 1)$:

$$\begin{aligned}\frac{dy}{dx} &= 2\operatorname{csch}(x^2 + 1)\frac{d}{dx}[\operatorname{csch}(x^2 + 1)] = 2\operatorname{csch}(x^2 + 1)[-\operatorname{csch}(x^2 + 1)\coth(x^2 + 1)\cdot 2x] \\ &= -4x\operatorname{csch}^2(x^2 + 1)\coth(x^2 + 1)\end{aligned}$$

9. $y = \frac{1}{4}\sinh 2x - \frac{1}{2}x$: $\quad \frac{dy}{dx} = \frac{1}{4}(\cosh 2x)2 - \frac{1}{2} = \frac{1}{2}(\cosh 2x - 1) = \sinh^2 x$

10. $y = \ln \tanh 2x$: $\quad \frac{dy}{dx} = \frac{1}{\tanh 2x}(2\operatorname{sech}^2 2x) = \frac{2}{\sinh 2x \cosh 2x} = 4\operatorname{csch} 4x$

11. Find the coordinates of the minimum point of the catenary $y = a\cosh \frac{x}{a}$.

$$f'(x) = \frac{1}{a}\left(a \sinh \frac{x}{a}\right) = \sinh \frac{x}{a} \qquad \text{and} \qquad f''(x) = \frac{1}{a}\cosh \frac{x}{a} = \frac{1}{a}\,\frac{e^{x/a} + e^{-x/a}}{2}$$

When $f'(x) = \dfrac{e^{x/a} - e^{-x/a}}{2} = 0$, $x = 0$; and $f''(0) > 0$. Hence, the point $(0, a)$ is the minimum point.

12. Examine (*a*) $y = \sinh x$, (*b*) $y = \cosh x$, and (*c*) $y = \tanh x$ for points of inflection.

(a) $f'(x) = \cosh x$, $f''(x) = \sinh x$, and $f'''(x) = \cosh x$.
$f''(x) = \sinh x = 0$ when $x = 0$, and $f'''(0) \neq 0$. Hence, the point $(0, 0)$ is a point of inflection.
(b) $f'(x) = \sinh x$, and $f''(x) = \cosh x \neq 0$ for all values of x. There is no point of inflection.
(c) $f'(x) = \operatorname{sech}^2 x$, $f''(x) = -2\operatorname{sech}^2 x \tanh x = -2\dfrac{\sinh x}{\cosh^3 x}$, and $f'''(x) = \dfrac{4\sinh^2 x - 2}{\cosh^4 x}$.
$f''(x) = 0$ when $x = 0$, and $f'''(0) \neq 0$. The point $(0, 0)$ is a point of inflection.

13. Derive: (a) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, for all x

(b) $\operatorname{sech}^{-1} x = \cosh^{-1}\dfrac{1}{x} = \ln\dfrac{1+\sqrt{1-x^2}}{x}$, for $0 < x \le 1$

(a) Let $\sinh^{-1} x = y$; then $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$ or, after multiplication by $2e^y$, $e^{2y} - 2xe^y - 1 = 0$. Solving for e^y yields $e^y = x + \sqrt{x^2+1}$, since $e^y > 0$. Thus, $y = \ln(x + \sqrt{x^2+1})$.

(b) Let $\operatorname{sech}^{-1} x = y$; then $x = \operatorname{sech} y = \dfrac{1}{\cosh y}$, so $\cosh y = \dfrac{1}{x}$. Hence $y = \cosh^{-1}\dfrac{1}{x} = \operatorname{sech}^{-1} x$. Also, $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}}$, from which $e^{2y}x - 2e^y + x = 0$.
Solving for e^y yields $e^y = \dfrac{1+\sqrt{1-x^2}}{x}$ for $y \ge 0$. Thus, $y = \ln\dfrac{1+\sqrt{1-x^2}}{x}$, $0 < x \le 1$.

14. Derive $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$.

Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and differentiation yields $\cosh y \dfrac{dy}{dx} = 1$; so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

In Problems 15 to 19, find dy/dx.

15. $y = \sinh^{-1} 3x$: $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{(3x)^2+1}}\dfrac{d}{dx}(3x) = \dfrac{3}{\sqrt{9x^2+1}}$

16. $y = \cosh^{-1} e^x$: $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{e^{2x}-1}}\dfrac{d}{dx}(e^x) = \dfrac{e^x}{\sqrt{e^{2x}-1}}$

17. $y = 2\tanh^{-1}(\tan \frac{1}{2}x)$: $\quad \dfrac{dy}{dx} = 2\dfrac{1}{1-\tan^2 \frac{1}{2}x}\dfrac{d}{dx}(\tan \frac{1}{2}x)$

$$= 2\frac{1}{1-\tan^2 \frac{1}{2}x}(\sec^2 \tfrac{1}{2}x)(\tfrac{1}{2}) = \frac{\sec^2 \frac{1}{2}x}{1-\tan^2 \frac{1}{2}x} = \sec x$$

18. $y = \coth^{-1}\dfrac{1}{x}$: $\quad \dfrac{dy}{dx} = \dfrac{1}{1-(1/x)^2}\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-1/x^2}{1-1/x^2} = \dfrac{-1}{x^2-1}$

19. $y = \operatorname{sech}^{-1}(\cos x)$: $\quad \dfrac{dy}{dx} = \dfrac{-1}{\cos x\sqrt{1-\cos^2 x}}\dfrac{d}{dx}(\cos x) = \dfrac{\sin x}{\cos x\sqrt{1-\cos^2 x}} = \sec x$

Supplementary Problems

20. (a) Sketch the curves of $y = e^x$ and $y = -e^{-x}$, and average the ordinates of the two curves for various values of x to obtain points on $y = \sinh x$. Complete the curve.
(b) Proceed as in (a), using $y = e^x$ and $y = e^{-x}$ to obtain the graph of $y = \cosh x$.

21. For the hyperbola $x^2 - y^2 = 1$ in Fig. 20-1, show that (*a*) $P(\cosh u, \sinh u)$ is a point on the hyperbola; (*b*) the tangent line at A intersects the line OP at $T(1, \tanh u)$.

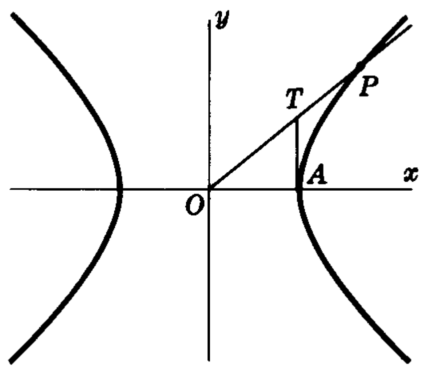


Fig. 20-1

22. Show: (*a*) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
(*b*) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
(*c*) $\sinh 2x = 2 \sinh x \cosh x$
(*d*) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$
(*e*) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

In Problems 23 to 28, find dy/dx.

23. $y = \sinh \frac{1}{4}x$ *Ans.* $\frac{1}{4}\cosh \frac{1}{4}x$

24. $y = \cosh^2 3x$ *Ans.* $3 \sinh 6x$

25. $y = \tanh 2x$ *Ans.* $2\,\mathrm{sech}^2\, 2x$

26. $y = \ln \cosh x$ *Ans.* $\tanh x$

27. $y = \arctan \sinh x$ *Ans.* $\mathrm{sech}\, x$

28. $y = \ln\sqrt{\tanh 2x}$ *Ans.* $2\,\mathrm{csch}\, 4x$

29. Show: (*a*) If $y = a \cosh \dfrac{x}{a}$, then $y'' = \dfrac{1}{a}\sqrt{1 + (y')^2}$.
(*b*) If $y = A \cosh bx + B \sinh bx$, where b, A, and B are constants, then $y'' = b^2 y$.

30. Show: (*a*) $\cosh^{-1} u = \ln(u + \sqrt{u^2 - 1})$, $u \ge 1$, and (*b*) $\tanh^{-1} u = \frac{1}{2} \ln \dfrac{1+u}{1-u}$, $u^2 < 1$.

31. (*a*) Trace the curve $y = \sinh^{-1} x$ by reflecting the curve $y = \sinh x$ in the 45° line.
(*b*) Trace the principal branch of $y = \cosh^{-1} x$ by reflecting the right half of $y = \cosh x$ in the 45° line.

32. Derive differentiation formulas 32 to 36, 38 to 40, and 42.

In Problems 33 to 36, find dy/dx.

33. $y = \sinh^{-1} \frac{1}{2}x$ *Ans.* $\dfrac{1}{\sqrt{x^2 + 4}}$

34. $y = \cosh^{-1} \dfrac{1}{x}$ *Ans.* $-\dfrac{1}{x\sqrt{1 - x^2}}$

35. $y = \tanh^{-1}(\sin x)$ *Ans.* $\sec x$

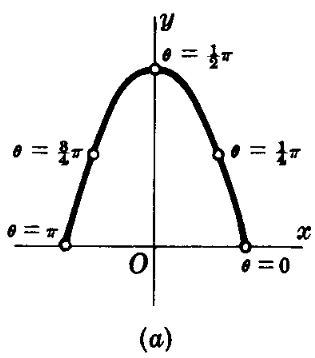
36. $x = a\,\mathrm{sech}^{-1} \dfrac{y}{a} - \sqrt{a^2 - y^2}$ *Ans.* $-\dfrac{y}{\sqrt{a^2 - y^2}}$

Chapter 21

Parametric Representation of Curves

PARAMETRIC EQUATIONS. If the coordinates (x, y) of a point P on a curve are given as functions $x = f(u)$, $y = g(u)$ of a third variable or *parameter* u, the equations $x = f(u)$ and $y = g(u)$ are called *parametric equations* of the curve.

EXAMPLE 1: (*a*) $x = \cos\theta$, $y = 4\sin^2\theta$ are parametric equations, with parameter θ, of the parabola $4x^2 + y = 4$, since $4x^2 + y = 4\cos^2\theta + 4\sin^2\theta = 4$.
(*b*) $x = \frac{1}{2}t$, $y = 4 - t^2$ is another parametric representation, with parameter t, of the same curve.
It should be noted that the first set of parametric equations represents only a portion of the parabola (Fig. 21-1(*a*)), whereas the second represents the entire curve (Fig. 21-1(*b*)).



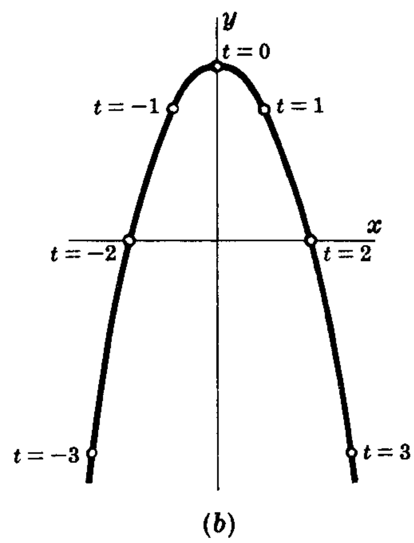


Fig. 21-1

EXAMPLE 2: (*a*) The equations $x = r\cos\theta$, $y = r\sin\theta$ represent the circle of radius r with center at the origin, since $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$. The parameter θ can be thought of as the angle from the positive x axis to the segment from the origin to the point P on the circle (Fig. 21-2).
(*b*) The equations $x = a + r\cos\theta$, $y = b + r\sin\theta$ represent the circle of radius r with center at (a, b), since $(x - a)^2 + (y - b)^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$.

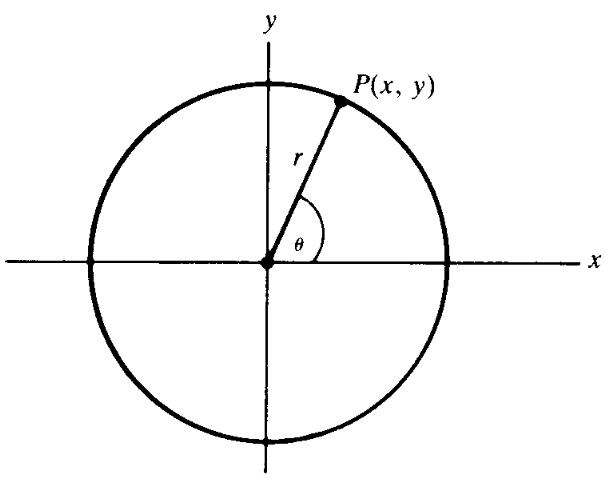


Fig. 21-2

THE FIRST DERIVATIVE $\frac{dy}{dx}$ is given by $\frac{dy}{dx} = \frac{dy/du}{dx/du}$.

THE SECOND DERIVATIVE $\frac{d^2y}{dx^2}$ is given by $\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$.

Solved Problems

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

$$\frac{dx}{d\theta} = 1 - \cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = \sin\theta\,. \quad \text{So} \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

Also,
$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)\frac{d\theta}{dx} = \frac{\cos\theta - 1}{(1-\cos\theta)^2}\,\frac{1}{1-\cos\theta} = -\frac{1}{(1-\cos\theta)^2}$$

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given $x = e^t\cos t$, $y = e^t\sin t$.

$$\frac{dx}{dt} = e^t(\cos t - \sin t) \qquad \frac{dy}{dt} = e^t(\sin t + \cos t) \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t + \cos t}{\cos t - \sin t}$$

Also,
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\sin t + \cos t}{\cos t - \sin t}\right)\frac{dx}{dt} = \frac{2}{(\cos t - \sin t)^2}\,\frac{1}{e^t(\cos t - \sin t)} = \frac{2}{e^t(\cos t - \sin t)^3}$$

3. Find the equation of the tangent to $x = \sqrt{t}$, $y = t - 1/\sqrt{t}$ at the point where $t = 4$.

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \quad \text{and} \quad \frac{dy}{dt} = 1 + \frac{1}{2t\sqrt{t}}\,. \quad \text{So} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2\sqrt{t} + \frac{1}{t}$$

At $t = 4$, $x = 2$, $y = 7/2$, and $m = dy/dx = 17/4$. The equation of the tangent is then $(y - 7/2) = (17/4)(x - 2)$ or $17x - 4y = 20$.

4. The position of a particle that is moving along a curve is given at time t by the parametric equations $x = 2 - 3\cos t$, $y = 3 + 2\sin t$, where x and y are measured in feet, and t in seconds. Find the time rate and direction of change of (a) the abscissa when $t = \pi/3$, (b) the ordinate when $t = 5\pi/3$, (c) θ, the angle of inclination of the tangent, when $t = 2\pi/3$. (See Fig. 21-3.)

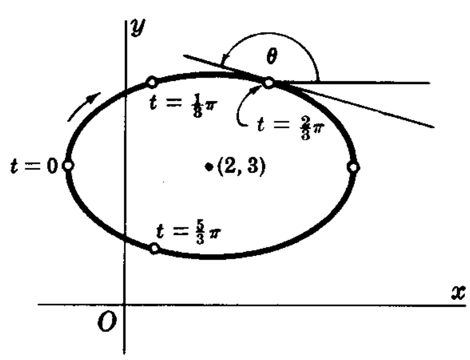


Fig. 21-3

$$\frac{dx}{dt} = 3\sin t \qquad \text{and} \qquad \frac{dy}{dt} = 2\cos t. \qquad \text{So} \qquad \tan\theta = \frac{dy}{dx} = \tfrac{2}{3}\cot t$$

(*a*) When $t = \pi/3$, $dx/dt = 3\sqrt{3}/2$. The abscissa is increasing at $3\sqrt{3}/2$ ft/sec.
(*b*) When $t = 5\pi/3$, $dy/dt = 2(\frac{1}{2}) = 1$. The ordinate is increasing at the rate 1 ft/sec.
(*c*) $\theta = \arctan(\frac{2}{3}\cot t)$, and $\dfrac{d\theta}{dt} = \dfrac{-6\csc^2 t}{9 + 4\cot^2 t}$. When $t = \dfrac{2\pi}{3}$, $\dfrac{d\theta}{dt} = \dfrac{-6(2/\sqrt{3})^2}{9 + 4(-1/\sqrt{3})^2} = -\dfrac{24}{31}$. The angle of inclination of the tangent is decreasing at a rate of $\frac{24}{31}$ rad/sec.

Supplementary Problems

In Problems 5 to 9, find (*a*) dy/dx and (*b*) d^2y/dx^2.

5. $x = 2 + t$, $y = 1 + t^2$ *Ans.* (*a*) $2t$; (*b*) 2

6. $x = t + 1/t$, $y = t + 1$ *Ans.* (*a*) $t^2/(t^2 - 1)$; (*b*) $-2t^3/(t^2 - 1)^3$

7. $x = 2\sin t$, $y = \cos 2t$ *Ans.* (*a*) $-2\sin t$; (*b*) -1

8. $x = \cos^3\theta$, $y = \sin^3\theta$ *Ans.* (*a*) $-\tan\theta$; (*b*) $1/(3\cos^4\theta\sin\theta)$

9. $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$ *Ans.* (*a*) $\tan\phi$; (*b*) $1/(a\phi\cos^3\phi)$

10. Find the slope of the curve $x = e^{-t}\cos 2t$, $y = e^{-2t}\sin 2t$ at the point $t = 0$. *Ans.* -2

11. Find the rectangular coordinates of the highest point of the curve $x = 96t$, $y = 96t - 16t^2$. (*Hint*: Find t for maximum y.) *Ans.* (288, 144)

12. Find the equation of the tangent and the normal to the curve (*a*) $x = 3e^t$, $y = 5e^{-t}$ at $t = 0$; (*b*) $x = a\cos^4\theta$, $y = a\sin^4\theta$ at $\theta = \frac{1}{4}\pi$.

Ans. (*a*) $5x + 3y - 30 = 0$, $3x - 5y + 16 = 0$; (*b*) $2x + 2y - a = 0$, $x - y = 0$

13. Find the equation of the tangent at any point $P(x, y)$ of the curve $x = a\cos^3 t$, $y = a\sin^3 t$. Show that the length of the segment of the tangent intercepted by the coordinate axes is a.

Ans. $x\sin t + y\cos t = \frac{1}{2}a\sin 2t$

14. For the curve $x = t^2 - 1$, $y = t^3 - t$, locate the points where the tangent line is (*a*) horizontal and (*b*) vertical. Show that at the point where the curve crosses itself, the two tangents are mutually perpendicular. *Ans.* (*a*) $t = \pm\sqrt{3}/3$; (*b*) $t = 0$

Chapter 22

Curvature

DERIVATIVE OF ARC LENGTH. Let $y = f(x)$ be a function having a continuous first derivative. Let A (see Fig. 22-1) be a fixed point on the graph, and denote by s the arc length measured from A to any other point on the curve. Let $P(x, y)$ be an arbitrary point, and $Q(x + \Delta x, y + \Delta y)$ a neighboring point on the curve. Denote by Δs the arc length from P to Q. The rate of change of s $(= AP)$ per unit change in x and its rate of change per unit change in y are given respectively by

$$\frac{ds}{dx} = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \frac{ds}{dy} = \lim_{\Delta y \to 0} \frac{\Delta s}{\Delta y} = \pm\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

The plus or minus sign is to be taken in the first formula according as s increases or decreases as x increases, and in the second formula according as s increases or decreases as y increases.

When a curve is given by the parametric equations $x = f(u)$, $y = g(u)$, the rate of change of s with respect to u is given by $\dfrac{ds}{du} = \pm\sqrt{\left(\dfrac{dx}{du}\right)^2 + \left(\dfrac{dy}{du}\right)^2}$. Here the plus or minus sign is to be taken according as s increases or decreases as u increases.

To avoid the repetition of ambiguous signs, we shall assume hereafter that direction on each arc has been established so that the derivative of arc length will be positive. (See Problems 1 to 5.)

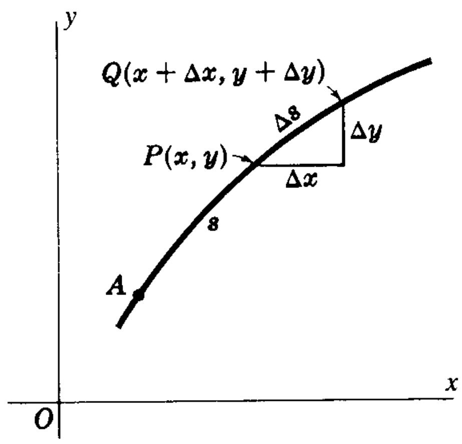


Fig. 22-1

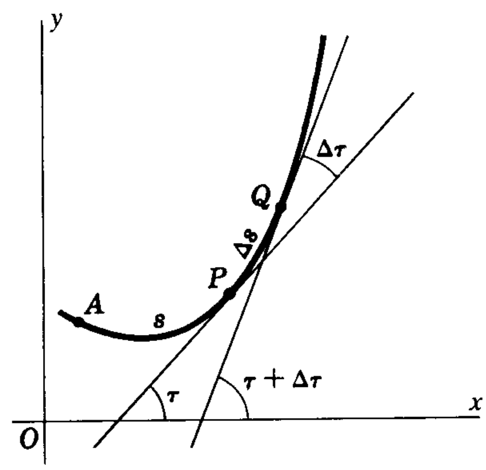


Fig. 22-2

CURVATURE. The curvature K of a curve $y = f(x)$, at any point P on it, is the rate of change in direction (i.e., of the angle of inclination τ of the tangent line at P) per unit of arc length s. (See Fig. 22-2.) Thus,

$$K = \frac{d\tau}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \tau}{\Delta s} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \qquad \text{or} \qquad K = \frac{-d^2x/dy^2}{[1 + (dx/dy)^2]^{3/2}} \tag{22.1}$$

From the first of these formulas, it is clear that K is positive when P is on an arc that is concave upward, and negative when P is on an arc that is concave downward.

K is sometimes defined so as to be positive, that is, as only the numerical values given by (*22.1*). With this latter definition, the sign of K in the answers below should be ignored.

THE RADIUS OF CURVATURE R for a point P on a curve is given by $R = |1/K|$, provided $K \neq 0$.

THE CIRCLE OF CURVATURE or *osculating circle* of a curve at a point P on it is the circle of radius R lying on the concave side of the curve and tangent to it at P (Fig. 22-3).

To construct the circle of curvature: On the concave side of the curve, construct the normal at P, and on it lay off $PC = R$. The point C is the center of the required circle.

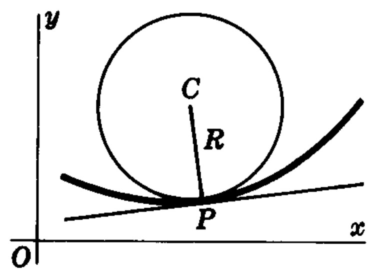


Fig. 22-3

THE CENTER OF CURVATURE for a point $P(x, y)$ of a curve is the center C of the circle of curvature at P. The coordinates (α, β) of the center of curvature are given by

$$\alpha = x - \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2} \qquad \beta = y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{d^2y/dx^2}$$

or by

$$\alpha = x - \frac{1 + \left(\dfrac{dx}{dy}\right)^2}{d^2x/dy^2} \qquad \beta = y - \frac{\dfrac{dx}{dy}\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]}{d^2x/dy^2}$$

THE EVOLUTE of a curve is the locus of the centers of curvature of the given curve. (See Problems 6 to 13.)

Solved Problems

1. Derive $\left(\dfrac{ds}{dx}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2$.

Refer to Fig. 22-1. On the curve $y = f(x)$, where $f(x)$ has a continuous derivative, let s denote the arc length from a fixed point A to a variable point $P(x, y)$. Denote by Δs the arc length from P to a neighboring point $Q(x + \Delta x, y + \Delta y)$ of the curve, and by PQ by the length of the chord joining P and Q. Now $\dfrac{\Delta s}{\Delta x} = \dfrac{\Delta s}{PQ}\dfrac{PQ}{\Delta x}$ and, since $(PQ)^2 = (\Delta x)^2 + (\Delta y)^2$,

$$\left(\frac{\Delta s}{\Delta x}\right)^2 = \left(\frac{\Delta s}{PQ}\right)^2\left(\frac{PQ}{\Delta x}\right)^2 = \left(\frac{\Delta s}{PQ}\right)^2\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2} = \left(\frac{\Delta s}{PQ}\right)^2\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

As Q approaches P along the curve, $\Delta x \to 0$, $\Delta y \to 0$, and $\dfrac{\Delta s}{PQ} = \dfrac{\text{arc } PQ}{\text{chord } PQ} \to 1$. (For a proof of the latter, see Problem 22 of Chapter 47.) Then

$$\left(\frac{ds}{dx}\right)^2 = \lim_{\Delta x \to 0}\left(\frac{\Delta s}{\Delta x}\right)^2 = \lim_{\Delta x \to 0}\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right] = 1 + \left(\frac{dy}{dx}\right)^2$$

2. Find ds/dx at $P(x, y)$ on the parabola $y = 3x^2$.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (6x)^2} = \sqrt{1 + 36x^2}$$

3. Find ds/dx and ds/dy at $P(x, y)$ on the ellipse $x^2 + 4y^2 = 8$.

Since $2x + 8y\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = -\dfrac{x}{4y}$ and $\dfrac{dx}{dy} = -\dfrac{4y}{x}$. Then

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{16y^2} = \frac{x^2 + 16y^2}{16y^2} = \frac{32 - 3x^2}{32 - 4x^2} \quad \text{and} \quad \frac{ds}{dx} = \sqrt{\frac{32 - 3x^2}{32 - 4x^2}}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{16y^2}{x^2} = \frac{x^2 + 16y^2}{x^2} = \frac{2 + 3y^2}{2 - y^2} \quad \text{and} \quad \frac{ds}{dy} = \sqrt{\frac{2 + 3y^2}{2 - y^2}}$$

4. Find $ds/d\theta$ at $P(\theta)$ on the curve $x = \sec\theta$, $y = \tan\theta$.

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{\sec^2\theta\tan^2\theta + \sec^4\theta} = |\sec\theta|\sqrt{\tan^2\theta + \sec^2\theta}$$

5. The coordinates (x, y) in feet of a moving particle P are given by $x = \cos t - 1$, $y = 2\sin t + 1$, where t is the time in seconds. At what rate is P moving along the curve when (*a*) $t = 5\pi/6$, (*b*) $t = 5\pi/3$, and (*c*) P is moving at its fastest and slowest?

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{1 + 3\cos^2 t}$$

(*a*) When $t = 5\pi/6$, $ds/dt = \sqrt{1 + 3(\frac{3}{4})} = \sqrt{13}/2$ ft/sec.

(*b*) When $t = 5\pi/3$, $ds/dt = \sqrt{1 + 3(\frac{1}{4})} = \sqrt{7}/2$ ft/sec.

(*c*) Let $S = \dfrac{ds}{dt} = \sqrt{1 + 3\cos^2 t}$. Then $\dfrac{dS}{dt} = \dfrac{-3\cos t\sin t}{S}$. Solving $dS/dt = 0$ gives the critical values $t = 0, \pi/2, \pi, 3\pi/2$.

When $t = 0$ and π, the rate $ds/dt = \sqrt{1 + 3(1)} = 2$ ft/sec is fastest. When $t = \pi/2$ and $3\pi/2$, the rate $ds/dt = \sqrt{1 + 3(0)} = 1$ ft/sec is slowest. The curve is shown in Fig. 22-4.

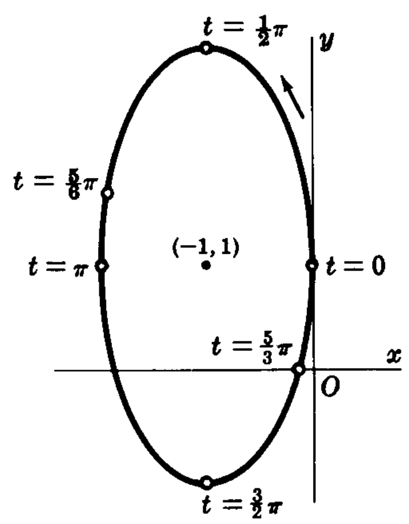


Fig. 22-4

6. Find the curvature of the parabola $y^2 = 12x$ at the points (*a*) $(3, 6)$; (*b*) $(\frac{3}{4}, -3)$; (*c*) $(0, 0)$.

$$\frac{dy}{dx} = \frac{6}{y}; \qquad \text{so} \qquad 1 + \left(\frac{dy}{dx}\right) = 1 + \frac{36}{y^2} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{6}{y^2}\frac{dy}{dx} = -\frac{36}{y^3}$$

(a) At $(3, 6)$: $1 + \left(\frac{dy}{dx}\right)^2 = 2$ and $\frac{d^2y}{dx^2} = -\frac{1}{6}$, so $K = \frac{-1/6}{2^{3/2}} = -\frac{\sqrt{2}}{24}$.

(b) At $(\frac{3}{4}, -3)$: $1 + \left(\frac{dy}{dx}\right)^2 = 5$ and $\frac{d^2y}{dx^2} = \frac{4}{3}$, so $K = \frac{4/3}{5^{3/2}} = \frac{4\sqrt{5}}{75}$.

(c) At $(0, 0)$, $\frac{dy}{dx}$ is undefined. But $\frac{dx}{dy} = \frac{y}{6} = 0$, $1 + \left(\frac{dx}{dy}\right)^2 = 1$, $\frac{d^2x}{dy^2} = \frac{1}{6}$, and $K = -\frac{1}{6}$.

7. Find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the highest point of an arch (see Fig. 22-5).

To find the highest point on the interval $0 < x < 2\pi$: $dy/d\theta = \sin\theta$, so that the critical value on the interval is $x = \pi$. Since $d^2y/d\theta^2 = \cos\theta < 0$ when $\theta = \pi$, the point $\theta = \pi$ is a relative maximum point and is the highest point of the curve on the interval.

To find the curvature,

$$\frac{dx}{d\theta} = 1 - \cos\theta \qquad \frac{dy}{d\theta} = \sin\theta \qquad \frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} \qquad \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1 - \cos\theta}\right)\frac{d\theta}{dx} = -\frac{1}{(1 - \cos\theta)^2}$$

At $\theta = \pi$, $dy/dx = 0$, $d^2y/dx^2 = -\frac{1}{4}$, and $K = -\frac{1}{4}$.

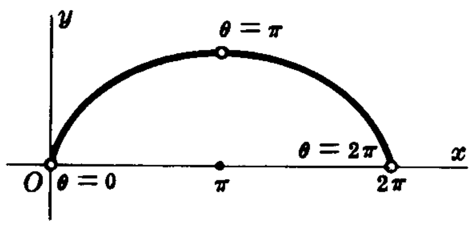


Fig. 22-5

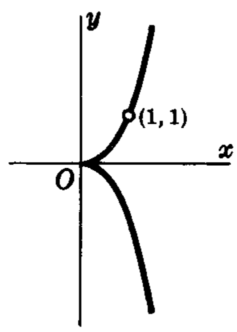


Fig. 22-6

8. Find the curvature of the cissoid $y^2(2 - x) = x^3$ at the point $(1, 1)$. (See Fig. 22-6).

Differentiating the given equation implicitly with respect to x, we obtain

$$-y^2 + (2 - x)2yy' = 3x^2 \tag{1}$$

and

$$-2yy' + (2 - x)2yy'' + (2 - x)2(y')^2 - 2yy' = 6x \tag{2}$$

From (1), for $x = y = 1$, $-1 + 2y' = 3$ and $y' = 2$. Similarly, from (2), for $x = y = 1$ and $y' = 2$, we find $y'' = 3$. Then $K = 3/(1 + 4)^{3/2} = 3\sqrt{5}/25$.

9. Find the point of greatest curvature on the curve $y = \ln x$.

$$\frac{dy}{dx} = \frac{1}{x} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2}. \qquad \text{So} \qquad K = \frac{-x}{(1 + x^2)^{3/2}} \qquad \text{and} \qquad \frac{dK}{dx} = \frac{2x^2 - 1}{(1 + x^2)^{5/2}}$$

The critical value is thus $x = 1/\sqrt{2}$. The required point is $(1/\sqrt{2}, -\frac{1}{2}\ln 2)$.

10. Find the coordinates of the center of curvature C of the curve $y = f(x)$ at a point $P(x, y)$ at which $y' \neq 0$. (See Fig. 22-3.)

The center of curvature $C(\alpha, \beta)$ lies (1) on the normal line at P and (2) at a distance R from P measured toward the concave side of the curve. These conditions give, respectively,

$$\beta - y = -\frac{1}{y'}(\alpha - x) \qquad \text{and} \qquad (\alpha - x)^2 + (\beta - y)^2 = R^2 = \frac{[1+(y')^2]^3}{(y'')^2}$$

From the first, $\alpha - x = -y'(\beta - y)$; substituting in the second yields

$$(\beta - y)^2[1+(y')^2] = \frac{[1+(y')^2]^3}{(y'')^2} \qquad \text{or} \qquad \beta - y = \pm\frac{1+(y')^2}{y''}$$

To determine the correct sign, note that when the curve is concave upward $y'' > 0$ and, since C then lies above P, $\beta - y > 0$. Thus, the proper sign in this case is $+$. (You should show that the sign is also $+$ when $y'' < 0$.) Thus,

$$\beta = y + \frac{1+(y')^2}{y''} \qquad \text{and} \qquad \alpha = x - \frac{y'[1+(y')^2]}{y''}$$

11. Find the equation of the circle of curvature of $2xy + x + y = 4$ at the point $(1, 1)$.

Differentiating yields $2y + 2xy' + 1 + y' = 0$. At $(1, 1)$, $y' = -1$ and $1 + (y')^2 = 2$. Differentiating again yields $4y' + 2xy'' + y'' = 0$. At $(1, 1)$, $y'' = \frac{4}{3}$. Then

$$K = \frac{4/3}{2\sqrt{2}} \qquad R = \frac{3\sqrt{2}}{2} \qquad \alpha = 1 - \frac{-1(2)}{4/3} = \frac{5}{2} \qquad \beta = 1 + \frac{2}{4/3} = \frac{5}{2}$$

The required equation is $(x-\alpha)^2 + (y-\beta)^2 = R^2$ or $(x - \frac{5}{2})^2 + (y - \frac{5}{2})^2 = \frac{9}{2}$.

12. Find the equation of the evolute of the parabola $y^2 = 12x$.

At $P(x, y)$:

$$\frac{dy}{dx} = \frac{6}{y} = \frac{\sqrt{3}}{\sqrt{x}} \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{36}{y^2} = 1 + \frac{3}{x} \qquad \frac{d^2y}{dx^2} = -\frac{36}{y^3} = -\frac{\sqrt{3}}{2x^{3/2}}$$

Then
$$\alpha = x - \frac{\sqrt{3/x}(1+3/x)}{-\sqrt{3}/2x^{3/2}} = x + \frac{2\sqrt{3}(x+3)}{\sqrt{3}} = 3x + 6$$

and
$$\beta = y + \frac{1 + 36/y^2}{-36/y^3} = y - \frac{y^3 + 36y}{36} = -\frac{y^3}{36}$$

The equations $\alpha = 3x + 6$, $\beta = -y^3/36$ may be regarded as parametric equations of the evolute with x and y, connected by the equation of the parabola, as parameters. However, it is relatively simple in this problem to eliminate the parameters. Thus, $x = (\alpha - 6)/3$, $y = -\sqrt[3]{36\beta}$, and substituting in the equation of the parabola, we have

$$(36\beta)^{2/3} = 4(\alpha - 6) \qquad \text{or} \qquad 81\beta^2 = 4(\alpha - 6)^3$$

The parabola and its evolute are shown in Fig. 22-7.

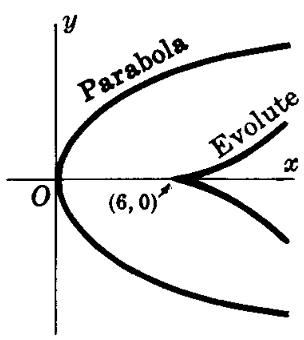


Fig. 22-7

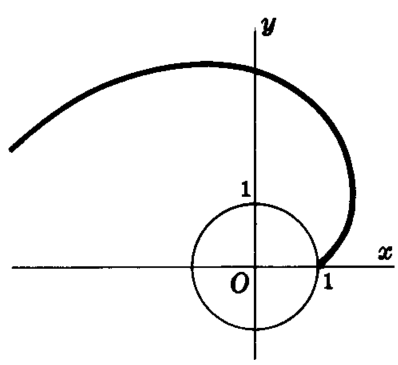


Fig. 22-8

13. Find the equation of the evolute of the curve $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$.

At $P(x, y)$:

$$\frac{dx}{d\theta} = \theta\cos\theta \qquad \frac{dy}{d\theta} = \theta\sin\theta \qquad \frac{dy}{dx} = \tan\theta \qquad \frac{d^2y}{dx^2} = \frac{\sec^2\theta}{\theta\cos\theta} = \frac{\sec^3\theta}{\theta}$$

Then
$$\alpha = x - \frac{\tan\theta\sec^2\theta}{(\sec^3\theta)/\theta} = x - \theta\sin\theta = \cos\theta$$

and
$$\beta = y + \frac{\sec^2\theta}{(\sec^3\theta)/\theta} = y + \theta\cos\theta = \sin\theta$$

and $\alpha = \cos\theta$, $\beta = \sin\theta$ are parametric equations of the evolute (see Fig. 22-8).

Supplementary Problems

In Problems 14 to 16, find ds/dx and ds/dy.

14. $x^2 + y^2 = 25$ *Ans.* $ds/dx = 5/\sqrt{25 - x^2}$, $ds/dy = 5/\sqrt{25 - y^2}$

15. $y^2 = x^3$ *Ans.* $ds/dx = \frac{1}{2}\sqrt{4 + 9x}$, $ds/dy = \sqrt{4 + 9y^{2/3}}/3y^{1/3}$

16. $x^{2/3} + y^{2/3} = a^{2/3}$ *Ans.* $ds/dx = (a/x)^{1/3}$, $ds/dy = (a/y)^{1/3}$

In Problems 17 to 19, find ds/dx.

17. $6xy = x^4 + 3$ *Ans.* $ds/dx = (x^4 + 1)/2x^2$

18. $27ay^2 = 4(x - a)^3$ *Ans.* $ds/dx = \sqrt{(x + 2a)/3a}$

19. $y = a\cosh x/a$ *Ans.* $ds/dx = \cosh x/a$

20. For the curve $x = f(u)$, $y = g(u)$, derive $(ds/du)^2 = (dx/du)^2 + (dy/du)^2$.

In Problems 21 to 24 find ds/dt.

21. $x = t^2$, $y = t^3$ *Ans.* $t\sqrt{4 + 9t^2}$

22. $x = \cos t$, $y = \sin t$ *Ans.* 1

23. $x = 2\cos t$, $y = 3\sin t$ *Ans.* $\sqrt{4 + 5\cos^2 t}$

24. $x = \cos^3 t$, $y = \sin^3 t$ *Ans.* $\frac{3}{2}\sin 2t$

25. Use $dy/dx = \tan\tau$ to obtain $dx/ds = \cos\tau$, $dy/ds = \sin\tau$.

26. Use $\tau = \arctan\left(\frac{dy}{dx}\right)$ to obtain $K = \frac{d\tau}{ds} = \frac{d\tau}{dx}\frac{dx}{ds} = \frac{y''}{\{1 + (y')^2\}^{3/2}}$.

27. Find the curvature of each curve at the given points.
(*a*) $y = x^3/3$ at $x = 0$, $x = 1$, $x = -2$ (*b*) $x^2 = 4ay$ at $x = 0$, $x = 2a$
(*c*) $y = \sin x$ at $x = 0$, $x = \frac{1}{2}\pi$ (*d*) $y = e^{-x^2}$ at $x = 0$

Ans. (*a*) 0, $\sqrt{2}/2$, $-4\sqrt{17}/289$; (*b*) $1/2a$, $\sqrt{2}/8a$; (*c*) 0, -1; (*d*) -2

28. Show that (*a*) the curvature of a straight line is zero and (*b*) the curvature of a circle is numerically the reciprocal of its radius.

29. Find the points of maximum curvature of (*a*) $y = e^x$, (*b*) $y = x^3/3$.

Ans. (*a*) $x = \frac{1}{2} \ln \frac{1}{2}$; (*b*) $x = 1/\sqrt[4]{5}$

30. Find the radius of curvature of (*a*) $x^3 + xy^2 - 6y^2 = 0$ at $(3, 3)$; (*b*) $x = a \operatorname{sech}^{-1} y/a - \sqrt{a^2 - y^2}$ at (x, y); (*c*) $x = 2a \tan \theta$, $y = a \tan^2 \theta$; (*d*) $x = a \cos^4 \theta$, $y = a \sin^4 \theta$.

Ans. (*a*) $5\sqrt{5}$; (*b*) $a\sqrt{a^2 - y^2}/|y|$; (*c*) $2a\,|\sec^3 \theta|$; (*d*) $2a(\sin^4 \theta + \cos^4 \theta)^{3/2}$

31. Find the center of curvature of (*a*) Problem 30(*a*); (*b*) $y = \sin x$ at a maximum point.

Ans. (*a*) $C(-7, 8)$; (*b*) $C(\frac{1}{2}\pi, 0)$

32. Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at the points $(0, 0)$ and $(3, 6)$.

Ans. $(x - 6)^2 + y^2 = 36$; $(x - 15)^2 + (y + 6)^2 = 288$

33. Find the equation of the evolute of (*a*) $b^2x^2 + a^2y^2 = a^2b^2$; (*b*) $x^{2/3} + y^{2/3} = a^{2/3}$; (*c*) $x = 2\cos t + \cos 2t$, $y = 2 \sin t + \sin 2t$.

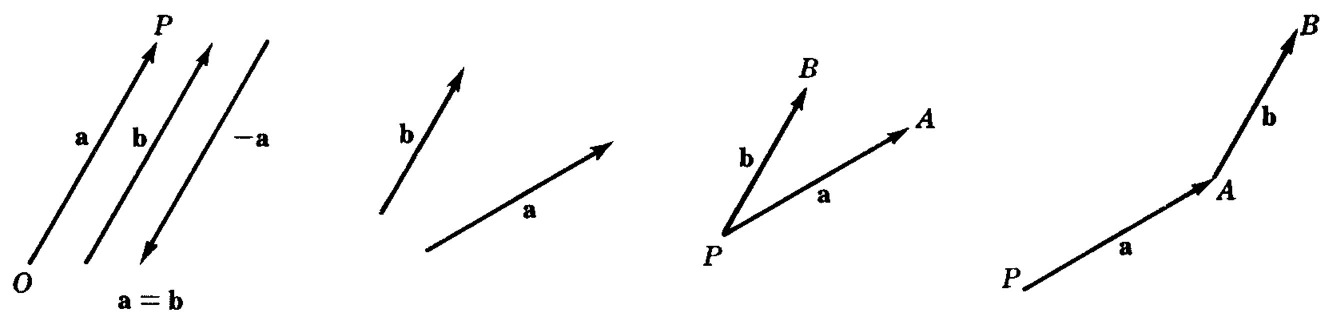
Ans. (*a*) $(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3}$; (*b*) $(\alpha + \beta)^{2/3} + (\alpha - \beta)^{2/3} = 2a^{2/3}$;
(*c*) $\alpha = \frac{1}{3}(2 \cos t - \cos 2t)$, $\beta = \frac{1}{3}(2 \sin t - \sin 2t)$

Chapter 23

Plane Vectors

SCALARS AND VECTORS. Quantities such as time, temperature, and speed, which have magnitude only, are called scalar quantities or *scalars*. Scalars, being merely numbers, obey all the laws of ordinary algebra; for example, 5 sec + 3 sec = 8 sec.

Quantities such as force, velocity, acceleration, and momentum, which have both magnitude and direction, are called vector quantities or *vectors*. Vectors are represented geometrically by directed line segments (arrows). The direction of the arrow (the angle which it makes with some fixed line of the plane) is the direction of the vector, and the length of the arrow (in terms of a chosen unit of measure) represents the magnitude of the vector. Scalars will be denoted here by letters $a, b, c, \ldots$ in ordinary type; vectors will be denoted in bold type by letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ or $\mathbf{OP}$ (see Fig. 23-1(a)). The magnitude of a vector $\mathbf{a}$ or $\mathbf{OP}$ will be denoted $|\mathbf{a}|$ or $|\mathbf{OP}|$.



(a) (b) (c) (d)

Fig. 23-1

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are called *equal* ($\mathbf{a} = \mathbf{b}$) if they have the same magnitude and the same direction. A vector whose magnitude is that of $\mathbf{a}$ but whose direction is opposite that of $\mathbf{a}$ is defined as the *negative* of $\mathbf{a}$ and is denoted $-\mathbf{a}$.

If $\mathbf{a}$ is a vector and k is a scalar, then $k\mathbf{a}$ is a vector whose direction is that of $\mathbf{a}$ and whose magnitude is k times that of $\mathbf{a}$ if k is positive, but whose direction is opposite that of $\mathbf{a}$ and whose magnitude is $|k|$ times that of $\mathbf{a}$ if k is negative.

Unless indicated otherwise, a given vector has no fixed position in the plane and so may be moved under parallel displacement at will. In particular, if $\mathbf{a}$ and $\mathbf{b}$ are two vectors (Fig. 23-1(b)), they may be placed so as to have a common initial or beginning point P (Fig. 23-1(c)) or so that the initial point of $\mathbf{b}$ coincides with the terminal or end point of $\mathbf{a}$ (Fig. 23-1(d)).

We also assume a *zero vector* $\mathbf{0}$ with magnitude 0 and no direction.

SUM AND DIFFERENCE OF TWO VECTORS. If $\mathbf{a}$ and $\mathbf{b}$ are the vectors of Fig. 23-1(b), their *sum* or *resultant* $\mathbf{a} + \mathbf{b}$ is found in either of two ways:

1. By placing the vectors as in Fig. 23-1(c) and completing the parallelogram $PAQB$ of Fig. 23-2(a). The vector $\mathbf{PQ}$ is the required sum.
2. By placing the vectors as in Fig. 23-1(d) and completing the triangle PAB of Fig. 23-2(b). Here, the vector $\mathbf{PB}$ is the required sum.

It follows from Fig. 23-2(b) that three vectors may be displaced to form a triangle provided one of them is either the sum or the negative of the sum of the other two.

If $\mathbf{a}$ and $\mathbf{b}$ are the vectors of Fig. 23-1(b), their difference $\mathbf{a} - \mathbf{b}$ is found in either of two ways:

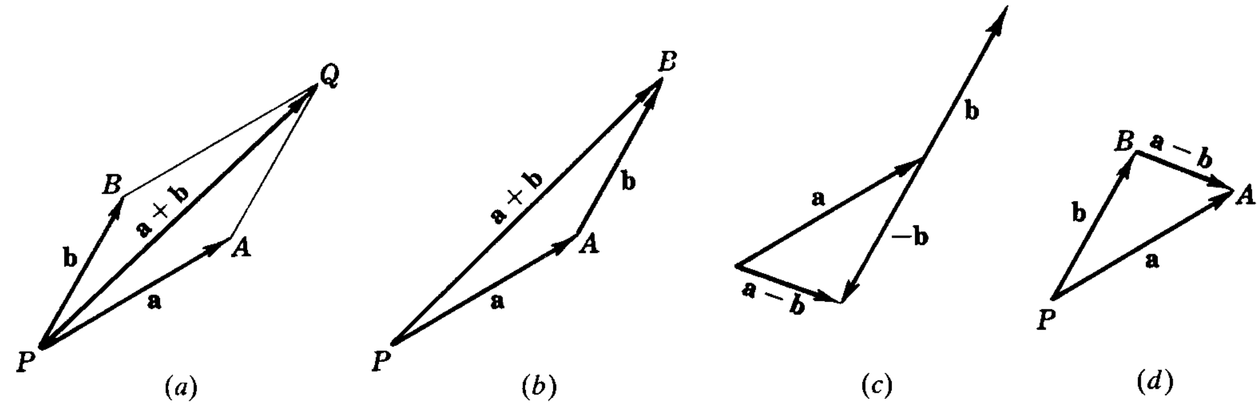


Fig. 23-2

1. From the relation $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$ as in Fig. 23-2(c).
2. By placing the vectors as in Fig. 23-1(c) and completing the triangle. In Fig. 23-2(d), the vector $\mathbf{BA}=\mathbf{a}-\mathbf{b}$.

If **a**, **b**, and **c** are vectors and k is a scalar, then

Property 23.1 (commutative law): $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$

Property 23.2 (associative law): $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$

Property 23.3 (distributive law): $k(\mathbf{a}+\mathbf{b})=k\mathbf{a}+k\mathbf{b}$

(See Problems 1 to 4.)

COMPONENTS OF A VECTOR. In Fig. 23-3(a), let $\mathbf{a}=\mathbf{PQ}$ be a given vector, and let PM and PN be any two other lines (directions) through P. Construct the parallelogram $PAQB$. Now

$$\mathbf{a}=\mathbf{PA}+\mathbf{PB}$$

and **a** is said to be *resolved* in the directions PM and PN. We shall call **PA** and **PB** the *vector components of* **a** *in the pair of directions PM and PN*.

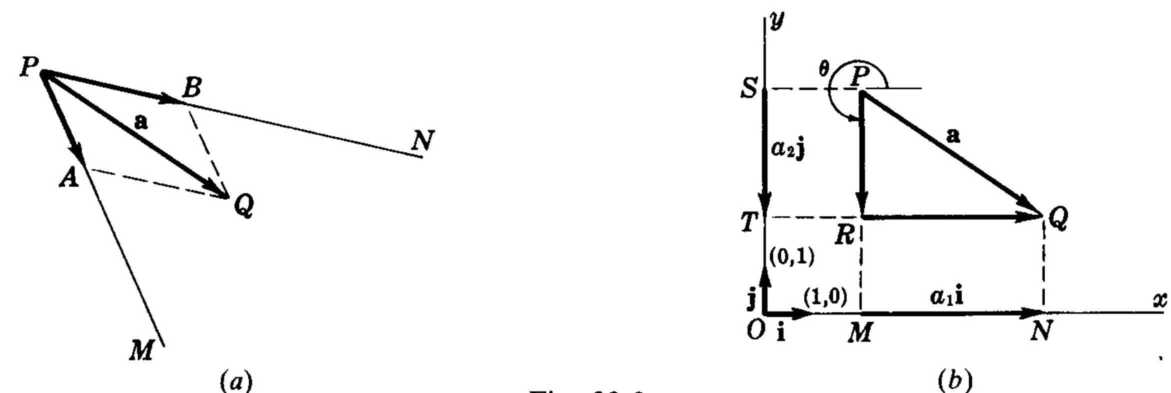


Fig. 23-3

Consider next the vector **a** in a rectangular coordinate system (Fig. 23-3(b)) having equal units of measure on the two axes. Denote by **i** the vector from $(0,0)$ to $(1,0)$, and by **j** the vector from $(0,0)$ to $(0,1)$. The direction of **i** is that of the positive x axis, the direction of **j** is that of the positive y axis, and both are *unit vectors*, that is, vectors of magnitude 1.

From the initial point P and the terminal point Q of **a**, drop perpendiculars to the x axis meeting it in M and N, respectively, and to the y axis meeting it in S and T, respectively. Now $\mathbf{MN}=a_1\mathbf{i}$, with a_1 positive, and $\mathbf{ST}=a_2\mathbf{j}$, with a_2 negative. Then $\mathbf{MN}=\mathbf{RQ}=a_1\mathbf{i}$, $\mathbf{ST}=\mathbf{PR}=a_2\mathbf{j}$, and

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j} \tag{23.1}$$

We shall call $a_1\mathbf{i}$ and $a_2\mathbf{j}$ the *vector components* of $\mathbf{a}$ (the pair of directions need not be mentioned), and the scalars a_1 and a_2 the *scalar components* or *x* and *y components* or simply *components* of $\mathbf{a}$. Note that the zero factor $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$.

Let the direction of $\mathbf{a}$ be given by the angle θ, for $0 \le \theta < 2\pi$, measured counterclockwise from the positive x axis to the vector. Then

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

and

$$\tan\theta = a_2/a_1$$

with the quadrant of θ being determined by

$$a_1 = |\mathbf{a}|\cos\theta \qquad a_2 = |\mathbf{a}|\sin\theta$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

Property 23.4: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$ and $a_2 = b_2$

Property 23.5: $k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j}$

Property 23.6: $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$

Property 23.7: $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$

(See Problem 5.)

SCALAR OR DOT PRODUCT. The scalar or dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \tag{23.4}$$

where θ is the smaller angle between the two vectors when they are drawn with a common initial point (see Fig. 23-4). We also let $\mathbf{a}\cdot\mathbf{0} = \mathbf{0}\cdot\mathbf{a} = 0$.

From (*23.4*) we have

Property 23.8 (commutative law): $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

Property 23.9: $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|\,|\mathbf{a}| = |\mathbf{a}|^2$ and $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$

Property 23.10: $\mathbf{a}\cdot\mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ is perpendicular to $\mathbf{b}$

Property 23.11: $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = 0$

Property 23.12: $\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j})\cdot(b_1\mathbf{i} + b_2\mathbf{j}) = a_1b_1 + a_2b_2$

Property 23.13 (distributive law): $\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

Property 23.14: $(\mathbf{a} + \mathbf{b})\cdot(\mathbf{c} + \mathbf{d}) = \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$

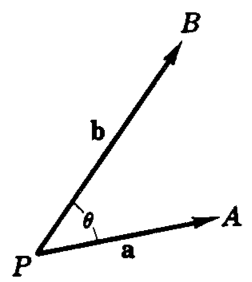


Fig. 23-4

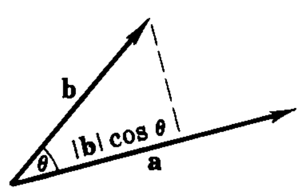


Fig. 23-5

SCALAR AND VECTOR PROJECTIONS. In (*23.1*), the scalar a_1 may be called the *scalar projection* of $\mathbf{a}$ on any vector whose direction is that of the positive x axis, while the vector $a_1\mathbf{i}$ may be called the *vector projection* of $\mathbf{a}$ on any vector whose direction is that of the positive x

axis. In Problem 7, the scalar projection $\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}$ and the vector projection $\left(\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}\right) \dfrac{\mathbf{b}}{|\mathbf{b}|}$ of a vector **a** on another vector **b** are found. (Note that when **b** has the direction of the positive x axis, then $\dfrac{\mathbf{b}}{|\mathbf{b}|} = \mathbf{i}$.)

There follows

Property 23.15: $\mathbf{a} \cdot \mathbf{b}$ is the product of the length of **a** and the scalar projection of **b** on **a**, or the product of the length of **b** and the scalar projection of **a** on **b**. (See Fig. 23-5.)

(See Problems 8 and 9.)

DIFFERENTIATION OF VECTORS. Let the curve of Fig. 23-6 be given by the parametric equations $x = f(u)$ and $y = g(u)$. The vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = \mathbf{i}f(u) + \mathbf{j}g(u)$$

joining the origin to the point $P(x, y)$ of the curve is called the *position vector* or radius vector of P. (Hereinafter, the letter **r** will be used exclusively to denote position vectors; thus, $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ is a "free" vector, while $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$ is the vector joining the origin to $P(3, 4)$.)

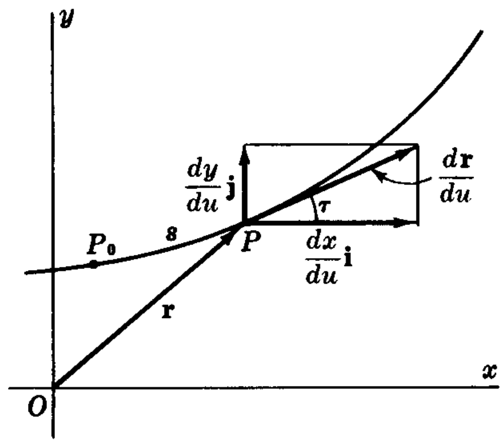


Fig. 23-6

The derivative of **r** with respect to u is given by

$$\frac{d\mathbf{r}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} \tag{23.5}$$

Let s denote the arc length measured from a fixed point P_0 of the curve so that s increases with u. If τ is the angle that $d\mathbf{r}/du$ makes with the positive x axis, then

$$\tan\tau = \frac{dy/du}{dx/du} = \frac{dy}{dx} = \text{slope of curve at } P$$

Moreover, $d\mathbf{r}/du$ is a vector of magnitude

$$\left|\frac{d\mathbf{r}}{du}\right| = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} = \frac{ds}{du}$$

whose direction is that of the tangent to the curve at P. It is customary to show this vector with P as initial point.

If now the scalar variable u is the length of arc s, (23.5) becomes

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \tag{23.6}$$

The direction of $\mathbf{t}$ is τ as before, while its magnitude is $\sqrt{(dx/ds)^2+(dy/ds)^2}=1$. Thus, $\mathbf{t}=d\mathbf{r}/ds$ is the *unit tangent* to the curve at P.

Since $\mathbf{t}$ is a unit vector, $\mathbf{t}$ and $d\mathbf{t}/ds$ are perpendicular (see Problem 11). Denote by $\mathbf{n}$ a unit vector at P having the direction of $d\mathbf{t}/ds$. As P moves along the curve shown in Fig. 23-7, the magnitude of $\mathbf{t}$ remains constant; hence, $d\mathbf{t}/ds$ measures the rate of change of the direction of $\mathbf{t}$. Thus, the magnitude of $d\mathbf{t}/ds$ at P is the numerical value of the curvature at P, that is, $|d\mathbf{t}/ds|=|K|$, and

$$\frac{d\mathbf{t}}{ds}=|K|\mathbf{n} \tag{23.7}$$

(See Problems 10 to 13.)

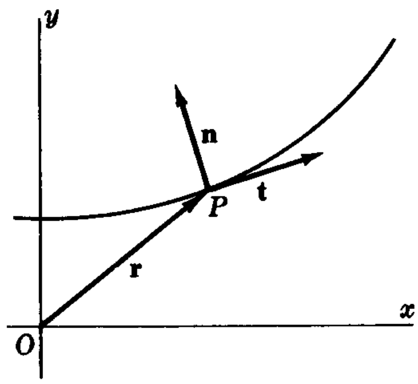


Fig. 23-7

Solved Problems

1. Prove $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

From Fig. 23-8, $\mathbf{a}+\mathbf{b}=\mathbf{PQ}=\mathbf{b}+\mathbf{a}$.

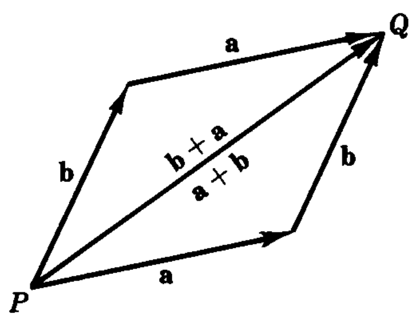


Fig. 23-8

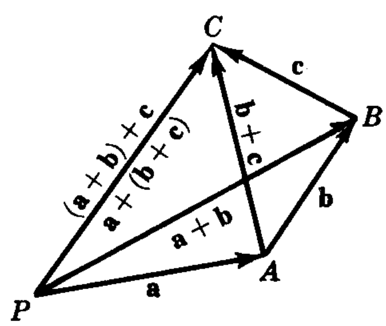


Fig. 23-9

2. Prove $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$

From Fig. 23-9, $\mathbf{PC}=\mathbf{PB}+\mathbf{BC}=(\mathbf{a}+\mathbf{b})+\mathbf{c}$. Also, $\mathbf{PC}=\mathbf{PA}+\mathbf{AC}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

3. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three vectors issuing from P such that their endpoints A, B, C lie on a line as shown in Fig. 23-10. If C divides BA in the ratio $x:y$ where $x+y=1$, show that $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$.

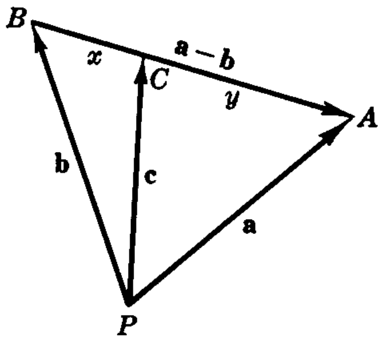


Fig. 23-10

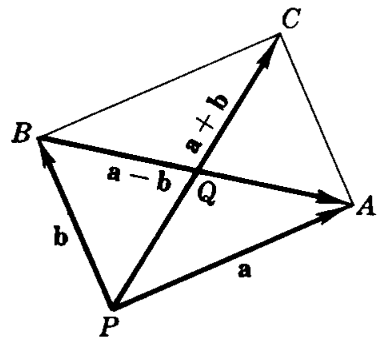


Fig. 23-11

$$\mathbf{c} = \mathbf{PB} + \mathbf{BC} = \mathbf{b} + x(\mathbf{a} - \mathbf{b}) = x\mathbf{a} + (1 - x)\mathbf{b} = x\mathbf{a} + y\mathbf{b}$$

For example, if C bisects BA, then $\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{BC} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$.

4. Prove: The diagonals of a parallelogram bisect each other.

Let the diagonals intersect at Q, as in Fig. 23-11. Since $\mathbf{PB} = \mathbf{PQ} + \mathbf{QB} = \mathbf{PQ} - \mathbf{BQ}$, there are positive numbers x and y such that $\mathbf{b} = x(\mathbf{a} + \mathbf{b}) - y(\mathbf{a} - \mathbf{b}) = (x - y)\mathbf{a} + (x + y)\mathbf{b}$. Then $x + y = 1$ and $x - y = 0$. Solving for x and y yields $x = y = \frac{1}{2}$, and Q is the midpoint of each diagonal.

5. For the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, find the magnitude and direction of (a) $\mathbf{a}$ and $\mathbf{b}$, (b) $\mathbf{a} + \mathbf{b}$, (c) $\mathbf{b} - \mathbf{a}$.

(a) For $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} = \sqrt{3^2 + 4^2} = 5$; $\tan\theta = a_2/a_1 = \frac{4}{3}$ and $\cos\theta = a_1/|\mathbf{a}| = \frac{3}{5}$; then θ is a first quadrant angle and is $53°8'$.
For $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$: $|\mathbf{b}| = \sqrt{4 + 1} = \sqrt{5}$; $\tan\theta = -\frac{1}{2}$ and $\cos\theta = 2/\sqrt{5}$; $\theta = 360° - 26°34' = 333°26'$.

(b) $\mathbf{a} + \mathbf{b} = (3\mathbf{i} + 4\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) = 5\mathbf{i} + 3\mathbf{j}$. Then $|\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + 3^2} = \sqrt{34}$. Since $\tan\theta = \frac{3}{5}$ and $\cos\theta = 5/\sqrt{34}$, $\theta = 30°58'$.

(c) $\mathbf{b} - \mathbf{a} = (2\mathbf{i} - \mathbf{j}) - (3\mathbf{i} + 4\mathbf{j}) = -\mathbf{i} - 5\mathbf{j}$. Then $|\mathbf{b} - \mathbf{a}| = \sqrt{26}$. Since $\tan\theta = 5$ and $\cos\theta = -1/\sqrt{26}$, $\theta = 258°41'$.

6. Prove: The median to the base of an isosceles triangle is perpendicular to the base. (In Fig. 23-12, $|\mathbf{a}| = |\mathbf{b}|$.)

From Problem 3, since $\mathbf{m}$ bisects the base,

$$\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

Then
$$\mathbf{m} \cdot (\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a})$$
$$= \tfrac{1}{2}(\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a}) = \tfrac{1}{2}(\mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a}) = 0$$

as was to be proved.

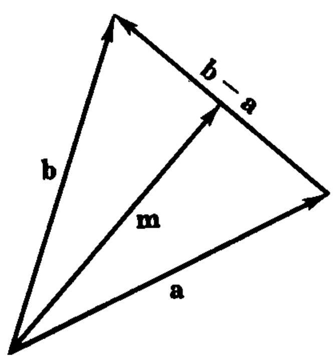


Fig. 23-12

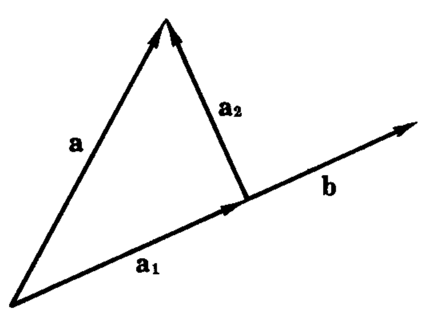


Fig. 23-13

7. Resolve a vector $\mathbf{a}$ into components $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively parallel and perpendicular to $\mathbf{b}$.

In Fig. 23-13, we have $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_1 = c\mathbf{b}$, and $\mathbf{a}_2 \cdot \mathbf{b} = 0$. These relations yield

$$\mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = \mathbf{a} - c\mathbf{b} \qquad \text{and} \qquad \mathbf{a}_2 \cdot \mathbf{b} = (\mathbf{a} - c\mathbf{b}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} - c|\mathbf{b}|^2 = 0 \qquad \text{or} \qquad c = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}$$

Thus, $\mathbf{a}_1 = c\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$ and $\mathbf{a}_2 = \mathbf{a} - c\mathbf{b} = \mathbf{a} - \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$.

The scalar $\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}$ is the scalar projection of $\mathbf{a}$ on $\mathbf{b}$; the vector $\left(\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}\right) \dfrac{\mathbf{b}}{|\mathbf{b}|}$ is the vector projection of $\mathbf{a}$ on $\mathbf{b}$.

8. Resolve $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ into components $\mathbf{a}_1$ and $\mathbf{a}_2$, parallel and perpendicular to $\mathbf{b} = 3\mathbf{i} + \mathbf{j}$.

From Problem 7, $c = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} = \dfrac{12+3}{10} = \dfrac{3}{2}$. Then $\mathbf{a}_1 = c\mathbf{b} = \frac{9}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$ and $\mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = -\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$.

9. Find the work done in moving an object along a vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ if the force applied is $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$.

$$\begin{aligned} \text{Work done} &= (\text{magnitude of } \mathbf{b} \text{ in the direction of } \mathbf{a})(\text{distance moved}) \\ &= (|\mathbf{b}| \cos\theta)|\mathbf{a}| = \mathbf{b} \cdot \mathbf{a} = (2\mathbf{i} + \mathbf{j}) \cdot (3\mathbf{i} + 4\mathbf{j}) = 10 \end{aligned}$$

10. If $\mathbf{a} = \mathbf{i}f_1(u) + \mathbf{j}f_2(u)$ and $\mathbf{b} = \mathbf{i}g_1(u) + \mathbf{j}g_2(u)$, show that $\dfrac{d}{du}(\mathbf{a} \cdot \mathbf{b}) = \dfrac{d\mathbf{a}}{du} \cdot \mathbf{b} + \mathbf{a} \cdot \dfrac{d\mathbf{b}}{du}$.

By Property 23.12, $\mathbf{a} \cdot \mathbf{b} = (\mathbf{i}f_1 + \mathbf{j}f_2) \cdot (\mathbf{i}g_1 + \mathbf{j}g_2) = f_1 g_1 + f_2 g_2$. Then

$$\begin{aligned} \frac{d}{du}(\mathbf{a} \cdot \mathbf{b}) &= f_1' g_1 + f_1 g_1' + f_2' g_2 + f_2 g_2' \qquad \left(f_1' = \frac{df_1(u)}{du}\right) \\ &= (f_1' g_1 + f_2' g_2) + (f_1 g_1' + f_2 g_2') \\ &= (\mathbf{i}f_1' + \mathbf{j}f_2') \cdot (\mathbf{i}g_1 + \mathbf{j}g_2) + (\mathbf{i}f_1 + \mathbf{j}f_2) \cdot (\mathbf{i}g_1' + \mathbf{j}g_2') = \frac{d\mathbf{a}}{du} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{du} \end{aligned}$$

11. If $\mathbf{a} = \mathbf{i}f_1(u) + \mathbf{j}f_2(u)$ is of constant magnitude, show that $\mathbf{a}$ and $d\mathbf{a}/du$ are perpendicular.

Since $|\mathbf{a}|$ is constant, $\mathbf{a} \cdot \mathbf{a} = \text{constant} \neq 0$, and we obtain, by Problem 10, $\dfrac{d}{du}(\mathbf{a} \cdot \mathbf{a}) = \dfrac{d\mathbf{a}}{du} \cdot \mathbf{a} + \mathbf{a} \cdot \dfrac{d\mathbf{a}}{du} = 2\mathbf{a} \cdot \dfrac{d\mathbf{a}}{du} = 0$. Then $\mathbf{a} \cdot \dfrac{d\mathbf{a}}{du} = 0$ so that $\mathbf{a}$ and $\dfrac{d\mathbf{a}}{du}$ are perpendicular.

Thus (as a geometric example), the tangent to a circle at one of its points P is perpendicular to the radius drawn to P.

12. Given $\mathbf{r} = \mathbf{i}\cos^2\theta + \mathbf{j}\sin^2\theta$, find $\mathbf{t}$.

$$\frac{d\mathbf{r}}{d\theta} = -\mathbf{i}\sin 2\theta + \mathbf{j}\sin 2\theta \qquad \text{and} \qquad \frac{ds}{d\theta} = \left|\frac{d\mathbf{r}}{d\theta}\right| = \sqrt{\frac{d\mathbf{r}}{d\theta} \cdot \frac{d\mathbf{r}}{d\theta}} = \sqrt{2}\sin 2\theta$$

Hence
$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

13. Given $x = a\cos^3\theta$, $y = a\sin^3\theta$, find $\mathbf{t}$ and $\mathbf{n}$ when $\theta = \frac{1}{4}\pi$.

We have $\mathbf{r} = a\mathbf{i}\cos^3\theta + a\mathbf{j}\sin^3\theta$. Then

$$\frac{d\mathbf{r}}{d\theta} = -3a\mathbf{i}\cos^2\theta\sin\theta + 3\alpha\mathbf{j}\sin^2\theta\cos\theta \qquad \text{and} \qquad \frac{ds}{d\theta} = \left|\frac{d\mathbf{r}}{d\theta}\right| = 3a\sin\theta\cos\theta$$

Hence $$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds} = -\mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

and $$\frac{d\mathbf{t}}{ds} = (\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)\frac{d\theta}{ds} = \frac{1}{3a\cos\theta}\mathbf{i} + \frac{1}{3a\sin\theta}\mathbf{j}$$

At $\theta = \frac{1}{4}\pi$: $\mathbf{t} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, $\frac{d\mathbf{t}}{ds} = \frac{\sqrt{2}}{3a}\mathbf{i} + \frac{\sqrt{2}}{3a}\mathbf{j}$, $|K| = \left|\frac{d\mathbf{t}}{ds}\right| = \frac{2}{3a}$, and $\mathbf{n} = \frac{1}{|K|}\frac{d\mathbf{t}}{ds} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$.

14. Show that the vector $\mathbf{a} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by + c = 0$.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points on the line. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$. Subtracting the first from the second yields

$$a(x_2 - x_1) + b(y_2 - y_1) = 0 \tag{1}$$

Now $$a(x_2 - x_1) + b(y_2 - y_1) = (a\mathbf{i} + b\mathbf{j})\cdot[(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}] = \mathbf{a}\cdot\mathbf{P_1P_2}$$

By (*1*), the left side is zero. Thus, **a** is perpendicular (normal) to the line.

15. Use vector methods to find:
(*a*) The equation of the line through $P_1(2, 3)$ and perpendicular to the line $x + 2y + 5 = 0$
(*b*) The equation of the line through $P_1(2, 3)$ and $P_2(5, -1)$

Take $P(x, y)$ to be any other point on the required line.

(*a*) By Problem 14, the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ is normal to $x + 2y + 5 = 0$. Then $\mathbf{P_1P} = (x-2)\mathbf{i} + (y-3)\mathbf{j}$ is parallel to **a** if

$$(x-2)\mathbf{i} + (y-3)\mathbf{j} = k(\mathbf{i} + 2\mathbf{j}) \quad (k \text{ a scalar})$$

Equating components, we have $x - 2 = k$ and $y - 3 = 2k$. Eliminating k, we obtain the required equation as $y - 3 = 2(x - 2)$ or $2x - y - 1 = 0$.

(*b*) We have $\mathbf{P_1P} = (x-2)\mathbf{i} + (y-3)\mathbf{j}$ and $\mathbf{P_1P_2} = 3\mathbf{i} - 4\mathbf{j}$

Now $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ is perpendicular to $\mathbf{P_1P_2}$ and, hence, to $\mathbf{P_1P}$. Thus, we may write

$$0 = \mathbf{a}\cdot\mathbf{P_1P} = (4\mathbf{i} + 3\mathbf{j})\cdot[(x-2)\mathbf{i} + (y-3)\mathbf{j}] \quad \text{or} \quad 4x + 3y - 17 = 0$$

16. Use vector methods to find the distance of the point $P_1(2, 3)$ from the line $3x + 4y - 12 = 0$.

At any convenient point on the line, say $A(4, 0)$, construct the vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ perpendicular to the line. The required distance is $d = |\mathbf{AP_1}|\cos\theta$ in Fig. 23-14. Now $\mathbf{a}\cdot\mathbf{AP_1} = |\mathbf{a}|\,|\mathbf{AP_1}|\cos\theta = |\mathbf{a}|\,d$; hence

$$d = \frac{\mathbf{a}\cdot\mathbf{AP_1}}{|\mathbf{a}|} = \frac{(3\mathbf{i} + 4\mathbf{j})\cdot(-2\mathbf{i} + 3\mathbf{j})}{5} = \frac{-6 + 12}{5} = \frac{6}{5}$$

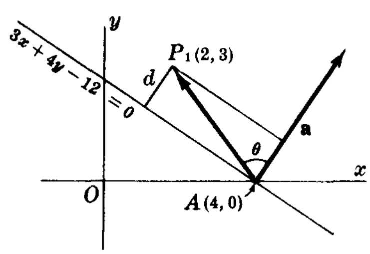


Fig. 23-14

Supplementary Problems

17. Given the vectors **a**, **b**, **c** in Fig. 23-15, construct (*a*) 2**a**; (*b*) -3**b**; (*c*) **a** + 2**b**; (*d*) **a** + **b** − **c**; (*e*) **a** − 2**b** + 3**c**.

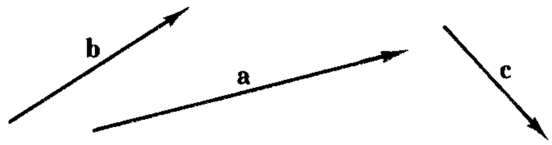


Fig. 23-15

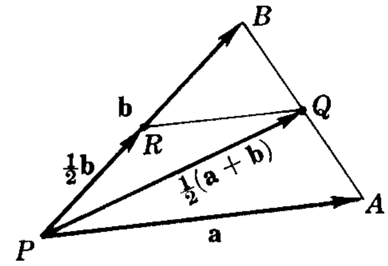


Fig. 23-16

18. Prove: The line joining the midpoints of two sides of a triangle is parallel to and one-half the length of the third side. (See Fig. 23-16.)

19. If **a**, **b**, **c**, **d** are consecutive sides of a quadrilateral (see Fig. 23-17), show that **a** + **b** + **c** + **d** = **0**. (*Hint*: Let P and Q be two nonconsecutive vertices.) Express **PQ** in two ways.

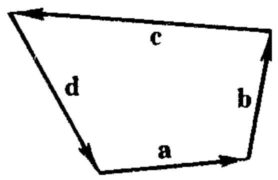


Fig. 23-17

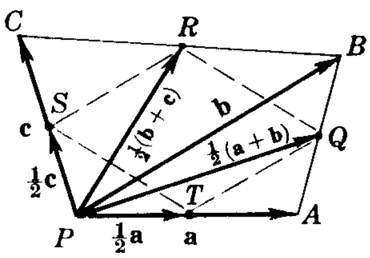


Fig. 23-18

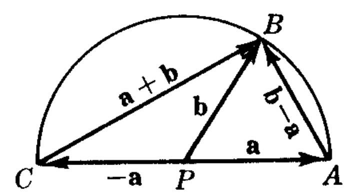


Fig. 23-19

20. Prove: If the midpoints of the consecutive sides of any quadrilateral are joined, the resulting quadrilateral is a parallelogram. (See Fig. 23-18.)

21. Using Fig. 23-19, in which $|\mathbf{a}| = |\mathbf{b}|$ is the radius of a circle, prove that the angle inscribed in a semicircle is a right angle.

22. Find the length of each of the following vectors and the angle it makes with the positive x axis: (*a*) $\mathbf{i}+\mathbf{j}$; (*b*) $-\mathbf{i}+\mathbf{j}$; (*c*) $\mathbf{i}+\sqrt{3}\mathbf{j}$; (*d*) $\mathbf{i}-\sqrt{3}\mathbf{j}$.

Ans. (*a*) $\sqrt{2}$, $\theta = \frac{1}{4}\pi$; (*b*) $\sqrt{2}$, $\theta = 3\pi/4$; (*c*) 2, $\theta = \pi/3$; (*d*) 2, $\theta = 5\pi/3$

23. Prove: If **u** is obtained by rotating the unit vector **i** counterclockwise about the origin through the angle θ, then $\mathbf{u} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$.

24. Use the law of cosines for triangles to obtain $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2)$.

25. Write each of the following vectors in the form $a\mathbf{i} + b\mathbf{j}$:
(*a*) The vector joining the origin to $P(2,-3)$
(*b*) The vector joining $P_1(2,3)$ to $P_2(4,2)$
(*c*) The vector joining $P_2(4,2)$ to $P_1(2,3)$
(*d*) The unit vector in the direction of $3\mathbf{i}+4\mathbf{j}$
(*e*) The vector having magnitude 6 and direction 120°

Ans. (*a*) $2\mathbf{i}-3\mathbf{j}$; (*b*) $2\mathbf{i}-\mathbf{j}$; (*c*) $-2\mathbf{i}+\mathbf{j}$; (*d*) $\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$; (*e*) $-3\mathbf{i}+3\sqrt{3}\mathbf{j}$

26. Using vector methods, derive the formula for the distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

27. Given $O(0,0)$, $A(3,1)$, and $B(1,5)$ as vertices of the parallelogram $OAPB$, find the coordinates of P.
Ans. $(4,6)$

28. (*a*) Find k so that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + k\mathbf{j}$ are perpendicular.
(*b*) Write a vector perpendicular to $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$.

29. Prove Properties 23.8 to 23.15.

30. Find the vector projection and scalar projection of $\mathbf{b}$ on $\mathbf{a}$, given: (*a*) $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} + \mathbf{j}$; (*b*) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 10\mathbf{i} + 2\mathbf{j}$. *Ans.* (*a*) $-\mathbf{i} + 2\mathbf{j}$, $-\sqrt{5}$; (*b*) $4\mathbf{i} + 6\mathbf{j}$, $2\sqrt{13}$

31. Prove: Three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ will, after parallel displacement, form a triangle provided (*a*) one of them is the sum of the other two or (*b*) $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

32. Show that $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$, and $\mathbf{c} = -7\mathbf{i} + 4\mathbf{j}$ are the sides of the right triangle. Verify that the midpoint of the hypotenuse is equidistant from the vertices.

33. Find the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds$, given: (*a*) $\mathbf{r} = 4\mathbf{i}\cos\theta + 4\mathbf{j}\sin\theta$; (*b*) $\mathbf{r} = e^{\theta}\mathbf{i} + e^{-\theta}\mathbf{j}$; (*c*) $\mathbf{r} = \theta\mathbf{i} + \theta^2\mathbf{j}$.
Ans. (*a*) $-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$; (*b*) $\dfrac{e^{\theta}\mathbf{i} - e^{-\theta}\mathbf{j}}{\sqrt{e^{2\theta} + e^{-2\theta}}}$; (*c*) $\dfrac{\mathbf{i} + 2\theta\mathbf{j}}{\sqrt{1 + 4\theta^2}}$

34. (*a*) Find $\mathbf{n}$ for the curve of Problem 33(*a*).
(*b*) Find $\mathbf{n}$ for the curve of Problem 33(*c*).
(*c*) Find $\mathbf{t}$ and $\mathbf{n}$ given $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$.

Ans. (*a*) $-\mathbf{i}\cos\theta - \mathbf{j}\sin\theta$; (*b*) $\dfrac{-2\theta}{\sqrt{1 + 4\theta^2}}\mathbf{i} + \dfrac{1}{\sqrt{1 + 4\theta^2}}\mathbf{j}$; (*c*) $\mathbf{t} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$, $\mathbf{n} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$

Chapter 24

Curvilinear Motion

VELOCITY IN CURVILINEAR MOTION. Consider a point $P(x, y)$ moving along a curve with the equations $x = f(t)$, $y = g(t)$, where t is time. By differentiating the position vector

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y \tag{24.1}$$

with respect to t, we obtain the *velocity vector*

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} = \mathbf{i}v_x + \mathbf{j}v_y \tag{24.2}$$

where $v_x = dx/dt$ and $v_y = dy/dt$.

The magnitude of $\mathbf{v}$ is called the *speed* and is given by

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_x^2 + v_y^2} = \frac{ds}{dt}$$

The direction of $\mathbf{v}$ at P is along the tangent to the path at P, as shown in Fig. 24-1. If τ denotes the direction of $\mathbf{v}$ (the angle between $\mathbf{v}$ and the positive x axis), then $\tan\tau = v_y/v_x$, with the quadrant being determined by $v_x = |\mathbf{v}|\cos\tau$ and $v_y = |\mathbf{v}|\sin\tau$.

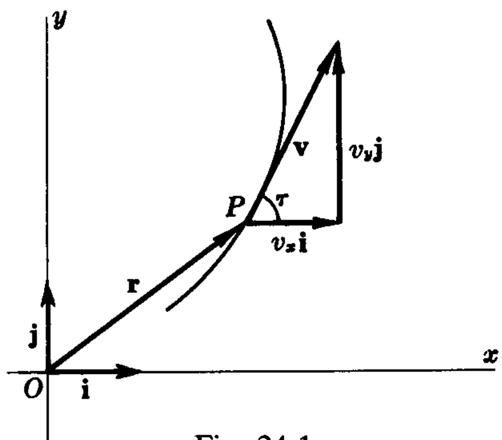


Fig. 24-1

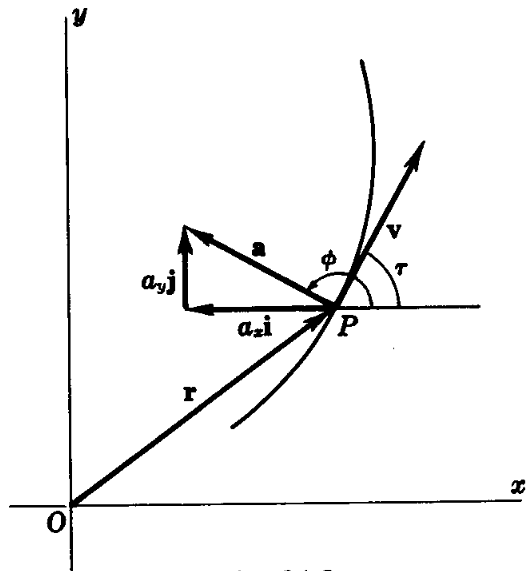


Fig. 24-2

ACCELERATION IN CURVILINEAR MOTION. Differentiating (*24.2*) with respect to t, we obtain the *acceleration vector*

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{i}\frac{d^2x}{dt^2} + \mathbf{j}\frac{d^2y}{dt^2} = \mathbf{i}a_x + \mathbf{j}a_y \tag{24.3}$$

where $a_x = d^2x/dt^2$ and $a_y = d^2y/dt^2$. The magnitude of $\mathbf{a}$ is given by

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_x^2 + a_y^2}$$

The direction ϕ of $\mathbf{a}$ is given by $\tan\phi = a_y/a_x$, with the quadrant being determined by $a_x = |\mathbf{a}|\cos\phi$ and $a_y = |\mathbf{a}|\sin\phi$. (See Fig. 24-2.)

In Problems 1 to 3, two solutions are offered. One uses the position vector (*24.1*), the velocity vector (*24.2*), and the acceleration vector (*24.3*). This solution requires a parametric representation of the path. The other and more popular solution makes use only of the x and y

components of these vectors; a parametric representation of the path is not necessary. The two solutions are, of course, basically the same.

TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION. By (*23.6*),

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{t}\frac{ds}{dt} \tag{24.4}$$

Then
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{t}\frac{d^2s}{dt^2} + \frac{d\mathbf{t}}{dt}\frac{ds}{dt} = \mathbf{t}\frac{d^2s}{dt^2} + \frac{d\mathbf{t}}{ds}\left(\frac{ds}{dt}\right)^2$$

$$= \mathbf{t}\frac{d^2s}{dt^2} + |K|\mathbf{n}\left(\frac{ds}{dt}\right)^2 \tag{24.5}$$

by (*23.7*).

Equation (*24.5*) resolves the acceleration vector at P along the tangent and normal there. Denoting the components by a_t and a_n, respectively, we have, for their magnitudes

$$|a_t| = \left|\frac{d^2s}{dt^2}\right| \quad \text{and} \quad |a_n| = \frac{(ds/dt)^2}{R} = \frac{|\mathbf{v}|^2}{R}$$

where R is the radius of curvature of the path at P. (See Fig. 24-3.)

Since $|\mathbf{a}|^2 = a_x^2 + a_y^2 = a_t^2 + a_n^2$, we have

$$a_n^2 = |\mathbf{a}|^2 - a_t^2$$

as a second means for determining $|a_n|$. (See Problems 4 to 8.)

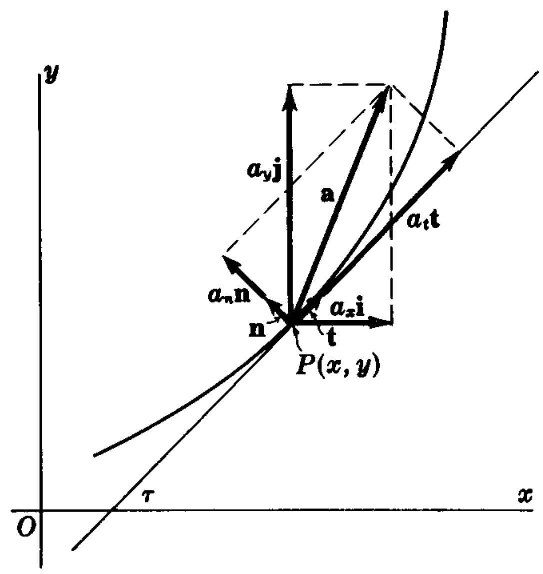


Fig. 24-3

Solved Problems

1. Discuss the motion given by the equations $x = \cos 2\pi t$, $y = 3\sin 2\pi t$. Find the magnitude and direction of the velocity and acceleration vectors when (*a*) $t = \frac{1}{6}$ and (*b*) $t = \frac{2}{3}$.

The motion is along the ellipse $9x^2 + y^2 = 9$. Beginning (at $t = 0$) at $(1, 0)$, the moving point traverses the curve counterclockwise.

First solution:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y = \mathbf{i}\cos 2\pi t + 3\mathbf{j}\sin 2\pi t$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}v_x + \mathbf{j}v_y = -2\pi\mathbf{i}\sin 2\pi t + 6\pi\mathbf{j}\cos 2\pi t$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i}a_x + \mathbf{j}a_y = -4\pi^2\mathbf{i}\cos 2\pi t - 12\pi^2\mathbf{j}\sin 2\pi t$$

(*a*) At $t = \frac{1}{6}$:

$$\mathbf{v} = -\sqrt{3}\pi\mathbf{i} + 3\pi\mathbf{j} \quad \text{and} \quad \mathbf{a} = -2\pi^2\mathbf{i} - 6\sqrt{3}\pi^2\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-\sqrt{3}\pi)^2 + (3\pi)^2} = 2\sqrt{3}\pi$$

$$\tan\tau = \frac{v_y}{v_x} = -\sqrt{3}, \qquad \cos\tau = \frac{v_x}{|\mathbf{v}|} = -\frac{1}{2}; \qquad \text{so} \qquad \tau = 120°$$

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{(-2\pi^2)^2 + (-6\sqrt{3}\pi^2)^2} = 4\sqrt{7}\pi^2$$

$$\tan\phi = \frac{a_y}{a_x} = 3\sqrt{3}, \qquad \cos\phi = \frac{a_x}{|\mathbf{a}|} = -\frac{1}{2\sqrt{7}}; \qquad \text{so} \qquad \phi = 259°6'$$

(*b*) At $t = \frac{2}{3}$:

$$\mathbf{v} = \sqrt{3}\pi\mathbf{i} - 3\pi\mathbf{j} \quad \text{and} \quad \mathbf{a} = 2\pi^2\mathbf{i} + 6\sqrt{3}\pi^2\mathbf{j}$$

$$|\mathbf{v}| = 2\sqrt{3}\pi, \qquad \tan\tau = -\sqrt{3}\cos\tau = \frac{1}{2}; \qquad \text{so} \qquad \tau = \frac{5\pi}{3}$$

$$|\mathbf{a}| = 4\sqrt{7}\pi^2, \qquad \tan\phi = 3\sqrt{3}\cos\phi = \frac{1}{2\sqrt{7}}; \qquad \text{so} \qquad \phi = 79°6'$$

Second solution:

$$x = \cos 2\pi t \qquad v_x = \frac{dx}{dt} = -2\pi\sin 2\pi t \qquad a_x = \frac{d^2x}{dt^2} = -4\pi^2\cos 2\pi t$$

$$y = 3\sin 2\pi t \qquad v_y = \frac{dy}{dt} = 6\pi\cos 2\pi t \qquad a_y = \frac{d^2y}{dt^2} = -12\pi^2\sin 2\pi t$$

(*a*) At $t = \frac{1}{6}$:

$$v_x = -\sqrt{3}\pi \qquad v_y = 3\pi \qquad |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = 2\sqrt{3}\pi$$

$$\tan\tau = \frac{v_y}{v_x} = -\sqrt{3}, \qquad \cos\tau = \frac{v_x}{|\mathbf{v}|} = -\frac{1}{2}; \qquad \text{so} \qquad \tau = 120°$$

$$a_x = -2\pi^2 \qquad a_y = -6\sqrt{3}\pi^2 \qquad |\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = 4\sqrt{7}\pi^2$$

$$\tan\phi = \frac{a_y}{a_x} = 3\sqrt{3}, \qquad \cos\phi = \frac{a_x}{|\mathbf{a}|} = -\frac{1}{2\sqrt{7}}; \qquad \text{so} \qquad \phi = 259°6'$$

(*b*) At $t = \frac{2}{3}$:

$$v_x = \sqrt{3}\pi \qquad v_y = -3\pi \qquad |\mathbf{v}| = 2\sqrt{3}\pi$$

$$\tan\tau = -\sqrt{3}, \qquad \cos\tau = \tfrac{1}{2}; \qquad \text{so} \qquad \tau = \frac{5\pi}{3}$$

$$a_x = 2\pi^2 \qquad a_y = 6\sqrt{3}\pi^2 \qquad |\mathbf{a}| = 4\sqrt{7}\pi^2$$

$$\tan\phi = 3\sqrt{3}, \qquad \cos\phi = \frac{1}{2\sqrt{7}}; \qquad \text{so} \qquad \phi = 79°6'$$

2. A point travels counterclockwise about the circle $x^2 + y^2 = 625$ at the rate $|\mathbf{v}| = 15$. Find τ, $|\mathbf{a}|$, and ϕ at (*a*) the point $(20, 15)$ and (*b*) the point $(5, -10\sqrt{6})$. Refer to Fig. 24-4.

First solution: We have

$$|\mathbf{v}|^2 = v_x^2 + v_y^2 = 225 \qquad (1)$$

and, by differentiation with respect to t,

$$v_x a_x + v_y a_y = 0 \qquad (2)$$

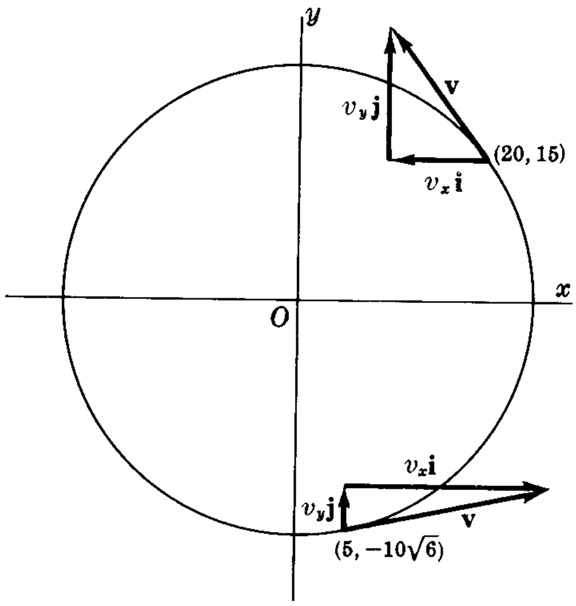


Fig. 24-4

From $x^2 + y^2 = 625$, we obtain by repeated differentiation

$$xv_x + yv_y = 0 \tag{3}$$

and
$$xa_x + v_x^2 + ya_y + v^2_y = 0$$

or
$$xa_x + ya_y = -225 \tag{4}$$

Solving (*1*) and (*3*) simultaneously, we have

$$v_x = \pm \tfrac{3}{5} y \tag{5}$$

Solving (*2*) and (*4*) simultaneously, we have

$$a_x = \frac{225 v_y}{yv_x - xv_y} \tag{6}$$

(*a*) From Fig. 24-4, $v_x < 0$ at (20, 15). From (*5*), $v_x = -9$; from (*3*), $v_y = 12$. Then $\tan \tau = -\frac{4}{3}$, $\cos \tau = -\frac{3}{5}$, and $\tau = 126°52'$. From (*6*), $a_x = -\frac{36}{5}$; from (*4*), $a_y = -\frac{27}{5}$; hence $|\mathbf{a}| = 9$. Then $\tan \phi = \frac{3}{4}$, $\cos \phi = -\frac{4}{5}$, and $\phi = 216°52'$.

(*b*) From the figure, $v_x > 0$ at $(5, -10/\sqrt{6})$. From (*5*), $v_x = 6\sqrt{6}$; from (*3*), $v_y = 3$. Then $\tan \tau = \sqrt{6}/12$, $\sin \tau = \frac{1}{5}$, and $\tau = 11°32'$. From (*6*), $a_x = -\frac{9}{5}$; from (*4*), $a_y = 18\sqrt{6}/5$; hence $|\mathbf{a}| = 9$. Then $\tan \phi = -2\sqrt{6}$, $\cos \phi = -\frac{1}{5}$, and $\phi = 101°32'$.

Second solution: Using the parametric equations $x = 25\cos\theta$, $y = 25\sin\theta$, we have at $P(x, y)$

$$\mathbf{r} = 25\mathbf{i}\cos\theta + 25\mathbf{j}\sin\phi$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-25\mathbf{i}\sin\theta + 25\mathbf{j}\cos\theta)\frac{d\theta}{dt} = -15\mathbf{i}\sin\theta + 15\mathbf{j}\cos\theta$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-15\mathbf{i}\cos\theta - 15\mathbf{j}\sin\theta)\frac{d\theta}{dt} = -9\mathbf{i}\cos\theta - 9\mathbf{j}\sin\theta$$

since $|\mathbf{v}| = 15$ is equivalent to a constant angular speed of $d\theta/dt = \frac{3}{5}$.

(*a*) At the point (20, 15), $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$. Thus,

$$\mathbf{v} = -9\mathbf{i} + 12\mathbf{j}, \quad \tan\tau = -\tfrac{4}{3}, \quad \cos\tau = -\tfrac{3}{5}; \quad \text{so} \quad \tau = 126°52'$$
$$\mathbf{a} = -\tfrac{36}{5}\mathbf{i} - \tfrac{27}{5}\mathbf{j}, \quad |\mathbf{a}| = 9, \quad \tan\phi = \tfrac{3}{4}, \quad \cos\phi = -\tfrac{4}{5}; \quad \text{so} \quad \phi = 216°52'$$

(*b*) At the point $(5, -10\sqrt{6})$, $\sin\theta = -\frac{2}{5}\sqrt{6}$ and $\cos\theta = \frac{1}{5}$. Thus,

$$\mathbf{v} = 6\sqrt{6}\mathbf{i} + 3\mathbf{j}, \quad \tan\tau = \sqrt{6}/12, \quad \cos\tau = \tfrac{2}{5}\sqrt{6}; \quad \text{so} \quad \tau = 11°32'$$
$$\mathbf{a} = -\tfrac{9}{5}\mathbf{i} + \tfrac{18}{5}\sqrt{6}\mathbf{j}, \quad |\mathbf{a}| = 9, \quad \tan\phi = -2\sqrt{6}, \quad \cos\phi = -\tfrac{1}{5}; \quad \text{so} \quad \phi = 101°32'$$

3. A particle moves on the first-quadrant arc of $x^2 = 8y$ so that $v_y = 2$. Find $|\mathbf{v}|$, τ, $|\mathbf{a}|$, and ϕ at the point $(4, 2)$.

First solution: Differentiating $x^2 = 8y$ twice with respect to t and using $v_y = 2$, we have

$$2xv_x = 8v_y = 16 \qquad \text{or} \qquad xv_x = 8 \qquad \text{and} \qquad xa_x + v_x^2 = 0$$

At $(4, 2)$: $\quad v_x = \dfrac{8}{x} = 2V\,, \quad |\mathbf{v}| = 2\sqrt{2}\,, \quad \tan\tau = 1\,, \quad \cos\tau = \tfrac{1}{2}\sqrt{2}\,; \quad \text{so} \quad \tau = \tfrac{1}{4}\pi$

$$a_x = -1\,, \quad a_y = 0\,, \quad |\mathbf{a}| = 1\,, \quad \tan\phi = 0\,, \quad \cos\phi = -1\,; \quad \text{so} \quad \phi = \pi$$

Second solution: Using the parametric equations $x = 4\theta$, $y = 2\theta^2$, we have

$$\mathbf{r} = 4\mathbf{i}\theta + 2\mathbf{j}\theta^2 \qquad \text{and} \qquad \mathbf{v} = 4\mathbf{i}\,\frac{d\theta}{dt} + 4\mathbf{j}\theta\,\frac{d\theta}{dt}$$

Since $v_y = 4\theta\,\dfrac{d\theta}{dt} = 2$ and $\dfrac{d\theta}{dt} = \dfrac{1}{2\theta}$, we have

$$\mathbf{v} = \frac{2}{\theta}\,\mathbf{i} + 2\mathbf{j} \qquad \text{and} \qquad \mathbf{a} = -\frac{1}{\theta^3}\,\mathbf{i}$$

At the point $(4, 2)$, $\theta = 1$. Then

$$\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}\,, \quad |\mathbf{v}| = 2\sqrt{2}\,, \quad \tan\tau = 1\,, \quad \cos\tau = \tfrac{1}{2}\sqrt{2}\,; \quad \text{so} \quad \tau = \tfrac{1}{4}\pi$$
$$\mathbf{a} = -\mathbf{i}\,, \quad |\mathbf{a}| = 1\,, \quad \tan\phi = 0\,, \quad \cos\phi = -1\,; \quad \text{so} \quad \phi = \pi$$

4. Find the magnitudes of the tangential and normal components of acceleration for the motion $x = e^t \cos t$, $y = e^t \sin t$ at any time t.

We have
$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y = \mathbf{i}e^t \cos t + \mathbf{j}e^t \sin t$$
$$\mathbf{v} = \mathbf{i}e^t(\cos t - \sin t) + \mathbf{j}e^t(\sin t + \cos t)$$
$$\mathbf{a} = -2\mathbf{i}e^t \sin t + 2\mathbf{j}e^t \cos t$$

Then $|\mathbf{a}| = 2e^t$. Also, $\dfrac{ds}{dt} = |\mathbf{v}| = \sqrt{2}e^t$ and $|a_t| = \left|\dfrac{d^2s}{dt^2}\right| = \sqrt{2}e^t$. Finally, $|a_n| = \sqrt{|\mathbf{a}|^2 - a_t^2} = \sqrt{2}e^t$.

5. A particle moves from left to right along the parabola $y = x^2$ with constant speed 5. Find the magnitude of the tangential and normal components of the acceleration at $(1, 1)$.

Since the speed is constant, $|a_t| = \left|\dfrac{d^2s}{dt^2}\right| = 0$.

At $(1, 1)$, $y' = 2x = 2$ and $y'' = 2$. The radius of curvature at $(1, 1)$ is then $R = \dfrac{[1 + (y')^2]^{3/2}}{|y''|} = \dfrac{5\sqrt{5}}{2}$.

Hence $|a_n| = \dfrac{|\mathbf{v}|^2}{R} = 2\sqrt{5}$.

6. The centrifugal force F exerted by a moving particle of weight W (both in pounds) at a point in its path is $F = \dfrac{W}{g}\,|a_n|$. Find the centrifugal force exerted by a particle, weighing 5 lb, at the ends of the major and minor axes as it traverses the elliptical path $x = 20\cos t$, $y = 15\sin t$, the measurements being in feet and seconds. Use $g = 32\ \text{ft/sec}^2$

We have
$$\mathbf{r} = 20\mathbf{i}\cos t + 15\mathbf{j}\sin t$$
$$\mathbf{v} = -20\mathbf{i}\sin t + 15\mathbf{j}\cos t$$
$$\mathbf{a} = -20\mathbf{i}\cos t - 15\mathbf{j}\sin t$$

Then
$$\frac{ds}{dt} = |\mathbf{v}| = \sqrt{400\sin^2 t + 225\cos^2 t} \qquad \frac{d^2s}{dt^2} = \frac{175\sin t\cos t}{\sqrt{400\sin^2 t + 225\cos^2 t}}$$

At the ends of the major axis ($t = 0$ or $t = \pi$):

$$|\mathbf{a}| = 20 \qquad |a_t| = \left|\frac{d^2s}{dt^2}\right| = 0 \qquad |a_n| = \sqrt{20^2 - 0^2} = 20 \qquad F = \frac{5}{32}\,20 = 3\,\frac{1}{4}\ \text{lb}$$

At the ends of the minor axis ($t = \pi/2$ or $t = 3\pi/2$):

$$|\mathbf{a}| = 15 \qquad |a_t| = 0 \qquad |a_n| = 15 \qquad F = \frac{5}{32}\,15 = \frac{75}{32}\text{ lb}$$

7. Assuming the equations of motion of a projectile to be $x = v_0 t\cos\psi$, $y = v_0 t\sin\psi - \frac{1}{2}gt^2$, where v_0 is the initial velocity, ψ is the angle of projection, $g = 32$ ft/sec^2, and x and y are measured in feet and t in seconds, find: (a) the equation of motion in rectangular coordinates; (b) the range; (c) the angle of projection for maximum range; and (d) the speed and direction of the projectile after 5 sec of flight if $v_0 = 500$ ft/sec and $\psi = 45°$. (See Fig. 24-5.)

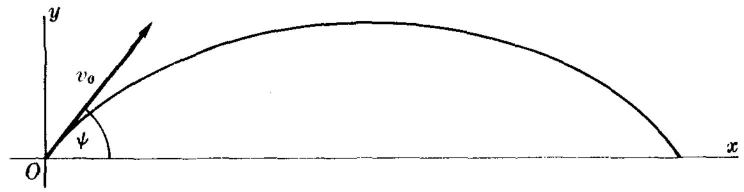


Fig. 24-5

(a) We solve the first of the equations for $t = \dfrac{x}{v_0\cos\psi}$ and substitute in the second:

$$y = v_0\,\frac{x}{v_0\cos\psi}\sin\psi - \tfrac{1}{2}g\left(\frac{x}{v_0\cos\psi}\right)^2 = x\tan\psi - \frac{gx^2}{2v_0^2\cos^2\psi}$$

(b) Solving $y = v_0 t\sin\psi - \frac{1}{2}gt^2 = 0$ for t, we get $t = 0$ and $t = (2v_0\sin\psi)/g$. For the latter, we have

$$\text{Range} = x = v_0\cos\psi\,\frac{2v_0\sin\psi}{g} = \frac{v_0^2\sin 2\psi}{g}$$

(c) For x a maximum, $\dfrac{dx}{d\psi} = \dfrac{2v_0^2\cos 2\psi}{g} = 0$; hence $\cos 2\psi = 0$ and $\psi = \frac{1}{4}\pi$.

(d) For $v_0 = 500$ and $\psi = \frac{1}{4}\pi$, $x = 250\sqrt{2}t$ and $y = 250\sqrt{2}t - 16t^2$. Then $v_x = 250\sqrt{2}$ and $v_y = 250\sqrt{2} - 32t$.

When $t = 5$, $v_x = 250\sqrt{2}$ and $v_y = 250\sqrt{2} - 160$. Then

$$\tan\tau = \frac{v_y}{v_x} = 0.5475\,. \qquad \text{So} \qquad \tau = 28°42' \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = 403\text{ ft/sec}$$

8. A point P moves on a circle $x = r\cos\beta$, $y = r\sin\beta$ with constant speed v. Show that, if the radius vector to P moves with angular velocity ω and angular acceleration α, (a) $v = r\omega$ and (b) $a = r\sqrt{\omega^4 + \alpha^2}$.

(a)
$$v_x = -r\sin\beta\,\frac{d\beta}{dt} = -r\omega\sin\beta \qquad \text{and} \qquad v_y = r\cos\beta\,\frac{d\beta}{dt} = r\omega\cos\beta$$

Then
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(r^2\sin^2\beta + r^2\cos^2\beta)\omega^2} = r\omega$$

(b)
$$a_x = \frac{dv_x}{dt} = -r\omega\cos\beta\,\frac{d\beta}{dt} - r\sin\beta\,\frac{d\omega}{dt} = -r\omega^2\cos\beta - r\alpha\sin\beta$$

$$a_y = \frac{dv_y}{dt} = -r\omega\sin\beta\,\frac{d\beta}{dt} + r\cos\beta\,\frac{d\omega}{dt} = -r\omega^2\sin\beta + r\alpha\cos\beta$$

Then
$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{r^2(\omega^4 + a^2)} = r\sqrt{\omega^4 + \alpha^2}$$

Supplementary Problems

9. Find the magnitude and direction of velocity and acceleration at time t, given
(a) $x = e^t$, $y = e^{2t} - 4e^t + 3$; at $t = 0$ *Ans.* (a) $|\mathbf{v}| = \sqrt{5}$, $\tau = 296°34'$; $|\mathbf{a}| = 1$, $\phi = 0$
(b) $x = 2 - t$, $y = 2t^3 - t$; at $t = 1$ *Ans.* (b) $|\mathbf{v}| = \sqrt{26}$, $\tau = 101°19'$; $|\mathbf{a}| = 12$, $\phi = \frac{1}{2}\pi$
(c) $x = \cos 3t$, $y = \sin t$; at $t = \frac{1}{4}\pi$ *Ans.* (c) $|\mathbf{v}| = \sqrt{5}$, $\tau = 161°34'$; $|\mathbf{a}| = \sqrt{41}$, $\phi = 353°40'$
(d) $x = e^t \cos t$, $y = e^t \sin t$; at $t = 0$ *Ans.* (d) $|\mathbf{v}| = \sqrt{2}$, $\tau = \frac{1}{4}\pi$; $|\mathbf{a}| = 2$, $\phi = \frac{1}{2}\pi$

10. A particle moves on the first-quadrant arc of the parabola $y^2 = 12x$ with $v_x = 15$. Find v_y, $|\mathbf{v}|$, and τ; and a_x, a_y, $|\mathbf{a}|$, and ϕ at $(3, 6)$.

Ans. $v_y = 15$, $|\mathbf{v}| = 15\sqrt{2}$, $\tau = \frac{1}{4}\pi$; $a_x = 0$, $a_y = -75/2$, $|\mathbf{a}| = 75/2$, $\phi = 3\pi/2$

11. A particle moves along the curve $y = x^3/3$ with $v_x = 2$ at all times. Find the magnitude and direction of the velocity and acceleration when $x = 3$. *Ans.* $|\mathbf{v}| = 2\sqrt{82}$, $\tau = 83°40'$; $|\mathbf{a}| = 24$, $\phi = \frac{1}{2}\pi$

12. A particle moves around a circle of radius 6 ft at the constant speed of 4 ft/sec. Determine the magnitude of its acceleration at any position. *Ans.* $|a_t| = 0$, $|\mathbf{a}| = |a_n| = 8/3$ ft/sec^2

13. Find the magnitude and direction of the velocity and acceleration, and the magnitudes of the tangential and normal components of acceleration at time t, for the motion
(a) $x = 3t$, $y = 9t - 3t^2$; at $t = 2$
(b) $x = \cos t + t \sin t$, $y = \sin t - t \cos t$; at $t = 1$.

Ans. (a) $|\mathbf{v}| = 3\sqrt{2}$, $\tau = 7\pi/4$; $|\mathbf{a}| = 6$, $\phi = 3\pi/2$; $|a_t| = |a_n| = 3\sqrt{2}$
(b) $|\mathbf{v}| = 1$, $\tau = 1$; $|\mathbf{a}| = \sqrt{2}$, $\phi = 102°18'$; $|a_t| = |a_n| = 1$

14. A particle moves along the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ so that $x = \frac{1}{2}t^2$, for $t > 0$. Find v_x, v_y, $|\mathbf{v}|$, and τ; a_x, a_y, $|\mathbf{a}|$, and ϕ; $|a_t|$ and $|a_n|$ when $t = 1$.

Ans. $v_x = 1$, $v_y = 0$, $|\mathbf{v}| = 1$, $\tau = 0$; $a_x = 1$, $a_y = 2$, $|\mathbf{a}| = \sqrt{5}$, $\phi = 63°26'$; $|a_t| = 1$, $|a_n| = 2$

15. A particle moves along the path $y = 2x - x^2$ with $v_x = 4$ at all times. Find the magnitudes of the tangential and normal components of acceleration at the position (a) $(1, 1)$ and (b) $(2, 0)$.

Ans. (a) $|a_t| = 0$, $|a_n| = 32$; (b) $|a_t| = 64/\sqrt{5}$, $|a_n| = 32/\sqrt{5}$

16. If a particle moves on a circle according to the equations $x = r\cos \omega t$, $y = r \sin \omega t$, show that its speed is ωr.

17. Prove that if a particle moves with constant speed, then its velocity and acceleration vectors are perpendicular; and, conversely, prove that if its velocity and acceleration vectors are perpendicular, then its speed is constant.

Chapter 25

Polar Coordinates

THE POSITION OF A POINT P in a given plane, relative to a fixed point O of the plane, may be described by giving the projections of the vector **OP** on two mutually perpendicular lines of the plane through O. This, in essence, is the rectangular coordinate system. Its position may also be described by giving the directed distance $\rho = OP$ and the angle θ which OP makes with a fixed half-line OX through O. This is the *polar coordinate system* (Fig. 25-1), in which point O is called the *pole*.

To each number pair (ρ, θ) there corresponds one and only one point. The converse is not true; for example, the point P in the figure may be described as $(\rho, \theta \pm 2n\pi)$ and $(-\rho, \theta \pm (2n+1)\pi)$, where n is any positive integer including 0. In particular, the polar coordinates of the pole may be given as $(0, \theta)$ with θ perfectly arbitrary.

The curve whose equation in polar coordinates is $\rho = f(\theta)$ or $F(\rho, \theta) = 0$ consists of the totality of distinct points (ρ, θ) that satisfy the equation.

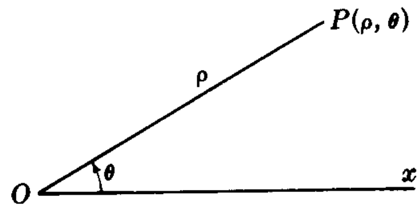


Fig. 25-1

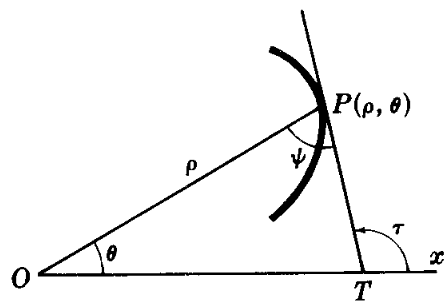


Fig. 25-2

THE ANGLE ψ from the radius vector OP to the tangent PT to a curve, at a point $P(\rho, \theta)$ on it, is given by

$$\tan \psi = \rho \frac{d\theta}{d\rho} = \frac{\rho}{\rho'} \qquad \text{where} \qquad \rho' = \frac{d\rho}{d\theta}$$

Tan ψ plays a role in polar coordinates somewhat similar to that of the slope of the tangent in rectangular coordinates. (See Problems 1 to 3.)

THE ANGLE OF INCLINATION τ of the tangent to a curve at a point $P(\rho, \theta)$ on it is given by

$$\tan \tau = \frac{\rho \cos \theta + \rho' \sin \theta}{-\rho \sin \theta + \cos \theta}$$

(See Problems 4 to 10.)

THE POINTS OF INTERSECTION of two curves whose equations are $\rho = f_1(\theta)$ and $\rho = f_2(\theta)$ may frequently be found by solving

$$f_1(\theta) = f_2(\theta) \tag{25.1}$$

EXAMPLE 1: Find the points of intersection of $\rho = 1 + \sin \theta$ and $\rho = 5 - 3 \sin \theta$.

Setting $1 + \sin \theta = 5 - 3 \sin \theta$, we have $\sin \theta = 1$. Then $\theta = \frac{1}{2}\pi$ and $(2, \frac{1}{2}\pi)$ is the only point of intersection.

Since a point may be represented by more than one pair of polar coordinates, the intersection of two curves may contain points for which no single pair of polar coordinates satisfies *(25.1)*.

EXAMPLE 2: Find the points of intersection of $\rho = 2\sin 2\theta$ and $\rho = 1$. Solution of the equation $2\sin 2\theta = 1$ yields $\sin 2\theta = \frac{1}{2}$ and $\theta = \pi/12,\ 5\pi/12,\ 13\pi/12,\ 17\pi/12$. We have found four points of intersection: $(1, \pi/12)$, $(1, 5\pi/12)$, $(1, 13\pi/12)$, and $(1, 17\pi/12)$.

But the circle $\rho = 1$ also can be represented as $\rho = -1$. Now solving $2\sin 2\theta = -1$, we obtain $\theta = 7\pi/12,\ 11\pi/12,\ 19\pi/12,\ 23\pi/12$ and the four additional points of intersection $(-1, 7\pi/12)$, $(-1, 11\pi/12)$, $(-1, 19\pi/12)$, $(-1, 23\pi/12)$.

When the pole is a point of intersection, it may not appear among the solutions of *(25.1)*. The pole is a point of intersection provided there are values of θ, say θ_1 and θ_2, such that $f_1(\theta_1) = 0$ and $f_2(\theta_2) = 0$.

EXAMPLE 3: Find the points of intersection of $\rho = \sin\theta$ and $\rho = \cos\theta$.

From the equation $\sin\theta = \cos\theta$, we obtain the point of intersection $(\frac{1}{2}\sqrt{2}, \frac{1}{4}\pi)$. The curves are, however, circles passing through the pole. But the pole is not obtained as a point of intersection from $\sin\theta = \cos\theta$, since on $\rho = \sin\theta$ it has coordinate $(0, 0)$ whereas on $\rho = \cos\theta$ it has coordinate $(0, \frac{1}{2}\pi)$.

EXAMPLE 4: Find the points of intersection of $\rho = \cos 2\theta$ and $\rho = \cos\theta$.

Setting $\cos 2\theta = 2\cos^2\theta - 1 = \cos\theta$, we find $(\cos\theta - 1)(2\cos\theta + 1) = 0$.

Then $\theta = 0,\ 2\pi/3,\ 4\pi/3$, and we have as points of intersection $(1, 0)$, $(-\frac{1}{2}, 2\pi/3)$, $(-\frac{1}{2}, 4\pi/3)$. The pole is also a point of intersection.

THE ANGLE OF INTERSECTION ϕ of two curves at a common point $P(\rho, \theta)$, not the pole, is given by

$$\tan\phi = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2}$$

where ψ_1 and ψ_2 are the angles from the radius vecor OP to the respective tangents to the curves at P (Fig. 25-3).

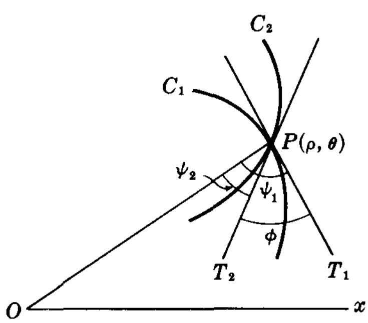


Fig. 25-3

The procedure for finding ϕ here is similar to that in the case of curves given in rectangular coordinates; the use of the tangents of the angles from the radius vector to the tangent instead of the slopes of the tangents is a matter of convenience in computing.

EXAMPLE 5. Find the (acute) angles of intersection of $\rho = \cos\theta$ and $\rho = \cos 2\theta$.

The points of intersection were found in Example 4. We also need ψ_1 and ψ_2: For $\rho = \cos\theta$, $\tan\psi_1 = -\cot\theta$; for $\rho = \cos 2\theta$, $\tan\psi_2 = -\frac{1}{2}\cot 2\theta$.

At the pole: On $\rho = \cos\theta$, the pole is given by $\theta = \pi/2$; on $\rho = \cos 2\theta$, the pole is given by $\theta = \pi/4$ and $3\pi/4$. Thus, at the pole there are two intersections, the acute angle being $\pi/4$ for each.

At the point $(1, 0)$: $\tan\psi_1 = -\cot 0 = \infty$ and $\tan\psi_2 = \infty$. Then $\psi_1 = \psi_2 = \pi/2$ and $\phi = 0$.

At the point $(-\frac{1}{2}, 2\pi/3)$: $\tan\psi_1 = \sqrt{3}/3$ and $\tan\psi_2 = -\sqrt{3}/6$. Then $\tan\phi = \dfrac{\sqrt{3}/3 + \sqrt{3}/6}{1 - 1/6} = 3\sqrt{3}/5$ and the acute angle of intersection is $\phi = 46°6'$.

By symmetry, this is also the acute angle of intersection at the point $(-\frac{1}{2}, 4\pi/3)$.

(See Problems 11 to 13.)

THE DERIVATIVE OF ARC LENGTH is given by $ds/d\theta = \sqrt{\rho^2 + (\rho')^2}$, where $\rho' = d\rho/d\theta$, and with the understanding that s increases as θ increases. (See Problems 14 to 16.)

THE CURVATURE of a curve is given by $K = \dfrac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[\rho^2 + (\rho')^2]^{3/2}}$. (See Problems 17 to 19.)

CURVILINEAR MOTION. Suppose as in Fig. 25-4, a particle P moves along a curve whose equation is given in polar coordinates as $\rho = f(\theta)$. If the curve is represented parametrically as

$$x = \rho\cos\theta = g(\theta) \qquad y = \rho\sin\theta = h(\theta)$$

then the position vector of P becomes

$$\mathbf{r} = \mathbf{OP} = x\mathbf{i} + y\mathbf{j} = \rho\mathbf{i}\cos\theta + \rho\mathbf{j}\sin\theta = \rho(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$$

and the motion may be studied as in Chapter 24.

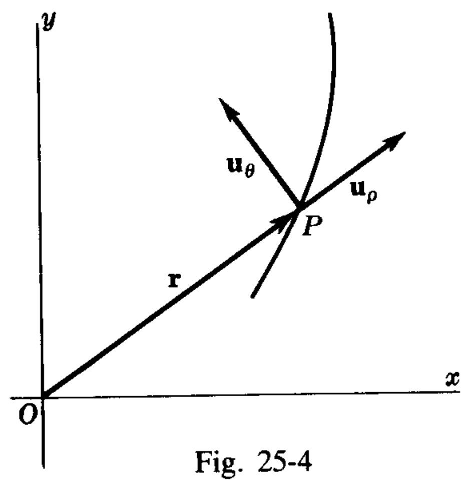


Fig. 25-4

An alternative procedure is to express **r** and, thus, **v** and **a** in terms of unit vectors along and perpendicular to the radius vector of P. For this purpose, we define the unit vector

$$\mathbf{u}_\rho = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

along **r** in the direction of increasing ρ, and the unit vector

$$\mathbf{u}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

perpendicular to **r** and in the direction of increasing θ. An easy calculation yields

$$\frac{d\mathbf{u}_\rho}{dt} = \frac{d\mathbf{u}_\rho}{d\theta}\frac{d\theta}{dt} = \mathbf{u}_\theta\frac{d\theta}{dt} \qquad \text{and} \qquad \frac{d\mathbf{u}_\theta}{dt} = -\mathbf{u}_\rho\frac{d\theta}{dt}$$

From
$$r = \rho\mathbf{u}_\rho$$

we obtain, in Problem 20,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{u}_\rho \frac{d\rho}{dt} + \rho \mathbf{u}_\theta \frac{d\theta}{dt} = v_\rho \mathbf{u}_\rho + v_\theta \mathbf{u}_\theta$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{u}_\rho \left[\frac{d^2\rho}{dt^2} - \rho \left(\frac{d\theta}{dt} \right)^2 \right] + \mathbf{u}_\theta \left[\rho \frac{d^2\theta}{dt^2} + 2 \frac{d\rho}{dt} \frac{d\theta}{dt} \right]$$
$$= a_\rho \mathbf{u}_\rho + a_\theta \mathbf{u}_\theta$$

Here $v_\rho = d\rho/dt$ and $v_\theta = \rho\, d\theta/dt$ are, respectively, the components of $\mathbf{v}$ along and perpendicular to the radius vector, and $a_\rho = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$ and $a_\theta = \rho \frac{d^2\theta}{dt^2} + 2 \frac{d\rho}{dt}\frac{d\theta}{dt}$ are the corresponding components of $\mathbf{a}$. (See Problem 21.)

Solved Problems

1. Derive $\tan \psi = \rho\, d\theta/d\rho$, where ψ is the angle measured from the radius vector OP of a point $P(\rho, \theta)$ on the curve of equation $\rho = f(\theta)$ to the tangent PT.

In Fig. 25-5, $Q(\rho + \Delta\rho, \theta + \Delta\theta)$ is a point on the curve near P. From the right triangle PSQ,

$$\tan \lambda = \frac{SP}{SQ} = \frac{SP}{OQ - OS} = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta} = \frac{\rho \sin \Delta\theta}{\rho(1 - \cos \Delta\theta) + \Delta\rho} = \frac{\rho \dfrac{\sin \Delta\theta}{\Delta\theta}}{\rho \dfrac{1 - \cos \Delta\theta}{\Delta\theta} + \dfrac{\Delta\rho}{\Delta\theta}}$$

Now as $Q \to P$ along the curve, $\Delta\theta \to 0$, $OQ \to OP$, $PQ \to PT$, and $\angle\lambda \to \angle\psi$.
As $\Delta\theta \to 0$, $\frac{\sin \Delta\theta}{\Delta\theta} \to 1$ and $\frac{1 - \cos \Delta\theta}{\Delta\theta} \to 0$ (see Chapter 17). Thus,

$$\tan \psi = \lim_{\Delta\theta \to 0} \tan \lambda = \frac{\rho}{d\rho/d\theta} = \rho \frac{d\theta}{d\rho}$$

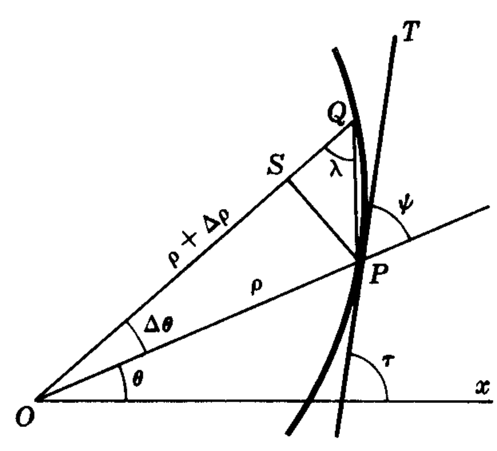


Fig. 25-5

In Problems 2 and 3, find $\tan \psi$ for the given curve at the given point.

2. $\rho = 2 + \cos\theta$; $\theta = \pi/3$. (See Fig. 25-6.)

At $\theta = \frac{\pi}{3}$: $\rho = 2 + \frac{1}{2} = \frac{5}{2}$, $\rho' = -\sin\theta = -\frac{\sqrt{3}}{2}$, and $\tan \psi = \frac{\rho}{\rho'} = -\frac{5}{\sqrt{3}}$.

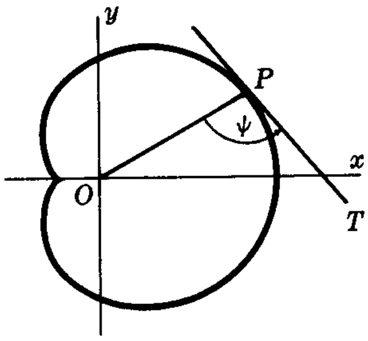


Fig. 25-6

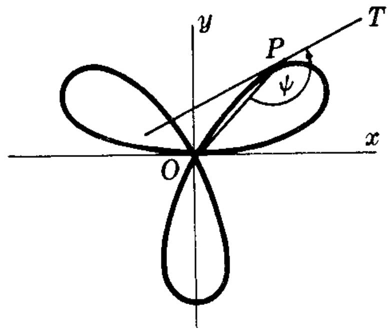


Fig. 25-7

3. $\rho = 2\sin 3\theta;\ \theta = \pi/4$. (See Fig. 25-7.)

At $\theta = \dfrac{\pi}{4}$: $\rho = 2\dfrac{1}{\sqrt{2}} = \sqrt{2}$, $\rho' = 6\cos 3\theta = 6\left(-\dfrac{1}{\sqrt{2}}\right) = -3\sqrt{2}$, and $\tan\psi = \dfrac{\rho}{\rho'} = -\dfrac{1}{3}$.

4. Derive $\tan\tau = \dfrac{\rho\cos\theta + \rho'\sin\theta}{-\rho\sin\theta + \rho'\cos\theta}$.

From Fig. 25-5, $\tau = \psi + \theta$ and

$$\tan\tau = \tan(\psi + \theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\tan\theta} = \frac{\rho\dfrac{d\theta}{d\rho} + \dfrac{\sin\theta}{\cos\theta}}{1 - \rho\dfrac{d\theta}{d\rho}\dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{\rho\cos\theta + \dfrac{d\rho}{d\theta}\sin\theta}{\dfrac{d\rho}{d\theta}\cos\theta - \rho\sin\theta} = \frac{\rho\cos\theta + \rho'\sin\theta}{-\rho\sin\theta + \rho'\cos\theta}$$

5. Show that if $\rho = f(\theta)$ passes through the pole and θ_1 is such that $f(\theta_1) = 0$, then the direction of the tangent to the curve at the pole $(0, \theta_1)$ is θ_1. (See Fig. 25-8.)

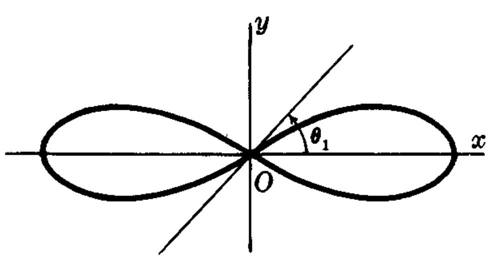


Fig. 25-8

At $(0, \theta_1)$, $\rho = 0$ and $\rho' = f'(\theta_1)$. If $\rho' \neq 0$, then

$$\tan\tau = \frac{\rho\cos\theta + \rho'\sin\theta}{-\rho\sin\theta + \rho'\cos\theta} = \frac{0 + f'(\theta_1)\sin\theta_1}{0 + f'(\theta_1)\cos\theta_1} = \tan\theta_1$$

If $\rho' = 0$,

$$\tan\tau = \lim_{\theta\to\theta_1}\frac{f'(\theta)\sin\theta}{f'(\theta)\cos\theta} = \tan\theta_1$$

In Problems 6 to 8, find the slope of the given curve at the given point.

6. $\rho = 1 - \cos\theta;\ \theta = \pi/2$. (See Fig. 25-9.)

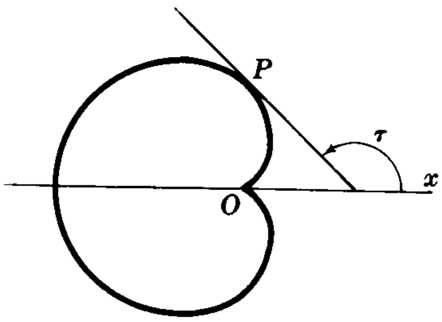


Fig. 25-9

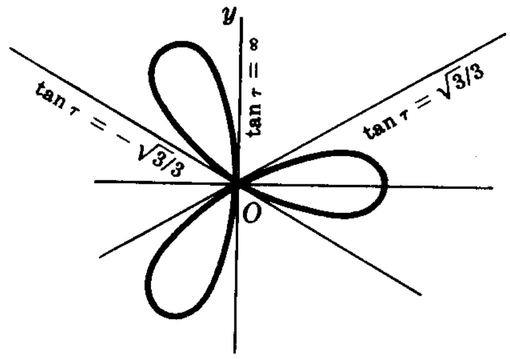


Fig. 25-10

At $\theta = \pi/2$: $\sin\theta = 1$, $\cos\theta = 0$, $\rho = 1$, $\rho' = \sin\theta = 1$, and

$$\tan\tau = \frac{\rho\cos\theta + \rho'\sin\theta}{-\rho\sin\theta + \rho'\cos\theta} = \frac{1\cdot 0 + 1\cdot 1}{-1\cdot 1 + 1\cdot 0} = -1$$

7. $\rho = \cos 3\theta$; pole. (See Fig. 25-10.)

When $\rho = 0$, $\cos 3\theta = 0$. Then $3\theta = \pi/2,\ 3\pi/2,\ 5\pi/2$, and $\theta = \pi/6,\ \pi/2,\ 5\pi/6$. By Problem 5, $\tan\tau = 1/\sqrt{3},\ \infty$, and $-1/\sqrt{3}$.

8. $\rho\theta = \alpha$; $\theta = \pi/3$.

At $\theta = \pi/3$: $\sin\theta = \sqrt{3}/2$, $\cos\theta = \frac{1}{2}$, $\rho = 3a/\pi$, and $\rho' = -a/\theta^2 = -9a/\pi^2$. Then

$$\tan\tau = \frac{\rho\cos\theta + \rho'\sin\theta}{-\rho\sin\theta + \rho'\cos\theta} = -\frac{\pi - 3\sqrt{3}}{\sqrt{3}\pi + 3}$$

9. Investigate $\rho = 1 + \sin\theta$ for horizontal and vertical tangents. (See Fig. 25-11.)

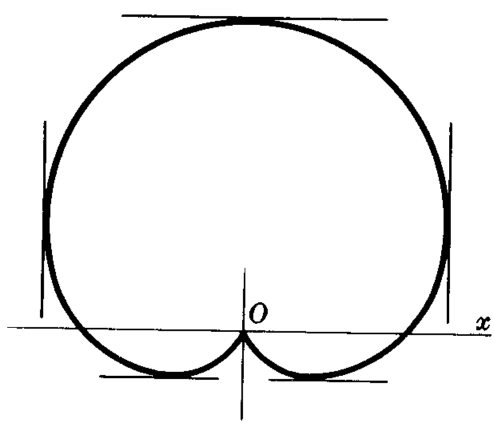


Fig. 25-11

At $P(\rho, \theta)$:

$$\tan\tau = \frac{(1+\sin\theta)\cos\theta + \cos\theta\sin\theta}{-(1+\sin\theta)\sin\theta + \cos^2\theta} = -\frac{\cos\theta\,(1 + 2\sin\theta)}{(\sin\theta + 1)(2\sin\theta - 1)}$$

We set $\cos\theta(1 + 2\sin\theta) = 0$ and solve, obtaining $\theta = \pi/2,\ 3\pi/2,\ 7\pi/6$, and $11\pi/6$. We also set $(\sin\theta + 1)(2\sin\theta - 1) = 0$ and solve, obtaining $\theta = 3\pi/2,\ \pi/6$, and $5\pi/6$.

For $\theta = \pi/2$: There is a horizontal tangent at $(2, \pi/2)$.

For $\theta = 7\pi/6$ and $11\pi/6$: There are horizontal tangents at $(1/2, 7\pi/6)$ and $(1/2, 11\pi/6)$.

For $\theta = \pi/6$ and $5\pi/6$: There are vertical tangents at $(3/2, \pi/6)$ and $(3/2, 5\pi/6)$.

For $\theta = 3\pi/2$: By Problem 5, there is a vertical tangent at the pole.

10. Show that the angle that the radius vector to any point of the cardioid $\rho = a(1 - \cos\theta)$ makes with the curve is one-half that which the radius vector makes with the polar axis.

At any point $P(\rho, \theta)$ on the cardioid, $\rho' = a\sin\theta$ and

$$\tan\psi = \frac{\rho}{\rho'} = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{1}{2}\theta; \qquad \text{so} \qquad \psi = \frac{1}{2}\theta$$

In Problems 11 to 13, find the angles of intersection of the given pair of curves.

11. $\rho = 3\cos\theta$, $\rho = 1 + \cos\theta$. (See Fig. 25-12.)

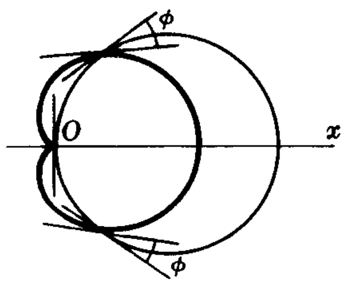


Fig. 25-12

Solve $3\cos\theta = 1 + \cos\theta$ for the points of intersection, obtaining $(3/2, \pi/3)$ and $(3/2, 5\pi/3)$. The curves also intersect at the pole.

For $\rho = 3\cos\theta$: $\quad \rho' = -3\sin\theta \quad$ and $\quad \tan\psi_1 = -\cot\theta$

For $\rho = 1 + \cos\theta$: $\quad \rho' = -\sin\theta \quad$ and $\quad \tan\psi_2 = -\dfrac{1+\cos\theta}{\sin\theta}$

At $\theta = \pi/3$, $\tan\psi_1 = -1/\sqrt{3}$, $\tan\psi_2 = -\sqrt{3}$, and $\tan\phi = 1/\sqrt{3}$. The acute angle of intersection at $(3/2, \pi/3)$ and, by symmetry, at $(3/2, 5\pi/3)$ is $\pi/6$.

At the pole, either a diagram or the result of Problem 5 shows that the curves are orthogonal.

12. $\rho = \sec^2\frac{1}{2}\theta$, $\rho = 3\csc^2\frac{1}{2}\theta$.

Solve $\sec^2\frac{1}{2}\theta = 3\csc^2\frac{1}{2}\theta$ for the points of intersection, obtaining $(4, 2\pi/3)$ and $(4, 4\pi/3)$.

For $\rho = \sec^2\frac{1}{2}\theta$: $\quad \rho' = \sec^2\frac{1}{2}\theta\tan\frac{1}{2}\theta \quad$ and $\quad \tan\psi_1 = \cot\frac{1}{2}\theta$

For $\rho = 3\csc^2\frac{1}{2}\theta$: $\quad \rho' = -3\csc^2\frac{1}{2}\theta\cot\frac{1}{2}\theta \quad$ and $\quad \tan\psi_2 = -\tan\frac{1}{2}\theta$

At $\theta = 2\pi/3$, $\tan\psi_1 = 1/\sqrt{3}$, $\tan\psi_2 = -\sqrt{3}$, and $\phi = \frac{1}{2}\pi$; the curves are orthogonal. Likewise, the curves are orthogonal at $\theta = 4\pi/3$.

13. $\rho = \sin 2\theta$, $\rho = \cos\theta$. (See Fig. 25-13.)

The curves intersect at the points $(\sqrt{3}/2, \pi/6)$ and $(-\sqrt{3}/2, 5\pi/6)$ and the pole.

For $\rho = \sin 2\theta$: $\quad \rho' = 2\cos 2\theta \quad$ and $\quad \tan\psi_1 = \frac{1}{2}\tan 2\theta$

For $\rho = \cos\theta$: $\quad \rho' = -\sin\theta \quad$ and $\quad \tan\psi_2 = -\cot\theta$

At $\theta = \pi/6$, $\tan\psi_1 = \sqrt{3}/2$, $\tan\psi_2 = -\sqrt{3}$, and $\tan\phi = -3\sqrt{3}$. The acute angle of intersection at the point $(\sqrt{3}/2, \pi/6)$ is $\phi = \arctan 3\sqrt{3} = 79°6'$. Similarly, at $\theta = 5\pi/6$, $\tan\psi_1 = -\sqrt{3}/2$, $\tan\psi_2 = \sqrt{3}$, and the angle of intersection is $\arctan 3\sqrt{3}$.

At the pole, the angles of intersection are $0°$ and $\pi/2$.

In Problems 14 to 16, find $ds/d\theta$ at the point $P(\rho, \theta)$.

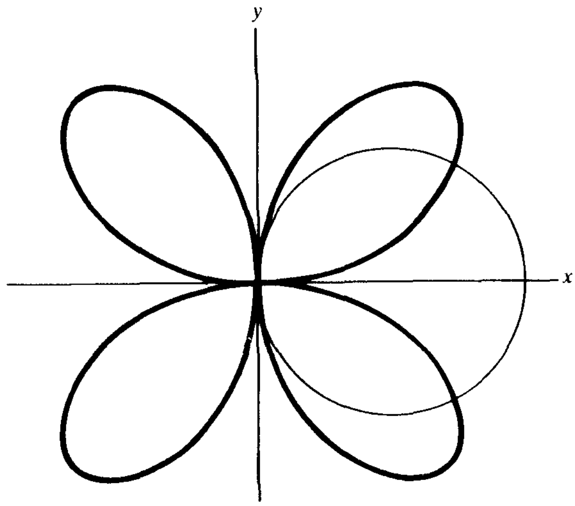


Fig. 25-13

14. $\rho = \cos 2\theta$.

$$\rho' = -2\sin 2\theta \qquad \text{and} \qquad \frac{ds}{d\theta} = \sqrt{\rho^2 + (\rho')^2} = \sqrt{\cos^2 2\theta + 4\sin^2 2\theta} = \sqrt{1 + 3\sin^2 2\theta}$$

15. $\rho(1 + \cos\theta) = 4$.

Differentiation yields $-\rho\sin\theta + \rho'(1 + \cos\theta) = 0$. Then

$$\rho' = \frac{\rho\sin\theta}{1 + \cos\theta} = \frac{4\sin\theta}{(1 + \cos\theta)^2} \qquad \text{and} \qquad \frac{ds}{d\theta} = \sqrt{\rho^2 + (\rho')^2} = \frac{4\sqrt{2}}{(1 + \cos\theta)^{3/2}}$$

16. $\rho = \sin^3 \frac{1}{3}\theta$. (Also evaluate $ds/d\theta$ at $\theta = \frac{1}{2}\pi$.)

$$\rho' = \sin^2 \tfrac{1}{3}\theta \cos \tfrac{1}{3}\theta \qquad \text{and} \qquad \frac{ds}{d\theta} = \sqrt{\sin^6 \tfrac{1}{3}\theta + \sin^4 \tfrac{1}{3}\theta \cos^2 \tfrac{1}{3}\theta} = \sin^2 \tfrac{1}{3}\theta$$

At $\theta = \frac{1}{2}\pi$, $ds/d\theta = \sin^2 \frac{1}{6}\pi = \frac{1}{4}$.

17. Derive $K = \dfrac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[\rho^2 + (\rho')^2]^{3/2}}$.

By definition, $K = d\tau/ds$. Now $\tau = \theta + \psi$ and

$$\frac{d\tau}{ds} = \frac{d\theta}{ds} + \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\psi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\psi}{d\theta}\right) \qquad \text{where} \qquad \psi = \arctan\frac{\rho}{\rho'}$$

Also,

$$\frac{d\psi}{d\theta} = \frac{[(\rho')^2 - \rho\rho'']/(\rho')^2}{1 + (\rho/\rho')^2} = \frac{(\rho')^2 - \rho\rho''}{\rho^2 + (\rho')^2}; \qquad \text{so} \qquad 1 + \frac{d\psi}{d\theta} = 1 + \frac{(\rho')^2 - \rho\rho''}{\rho^2 + (\rho')^2} = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{\rho^2 + (\rho')^2}$$

Thus,
$$K = \frac{d\theta}{ds}\left(1 + \frac{d\psi}{d\theta}\right) = \frac{1 + d\psi/d\theta}{ds/d\theta} = \frac{1 + d\psi/d\theta}{\sqrt{\rho^2 + (\rho')^2}} = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[\rho^2 + (\rho')^2]^{3/2}}$$

18. Let $\rho = 2 + \sin\theta$. Find the curvature at the point $P(\rho, \theta)$.

$$K = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[\rho^2 + (\rho')^2]^{3/2}} = \frac{(2 + \sin\theta)^2 + 2\cos^2\theta + (\sin\theta)(2 + \sin\theta)}{[(2 + \sin\theta)^2 + \cos^2\theta]^{3/2}} = \frac{6(1 + \sin\theta)}{(5 + 4\sin\theta)^{3/2}}$$

19. Let $\rho(1-\cos\theta)=1$. Find the curvature at $\theta=\pi/2$ and at $\theta=4\pi/3$.

$$\rho'=\frac{-\sin\theta}{(1-\cos\theta)^2} \quad\text{and}\quad \rho''=\frac{-\cos\theta}{(1-\cos\theta)^2}+\frac{2\sin^2\theta}{(1-\cos\theta)^3} \quad\text{so}\quad K=\sin^3\frac{\theta}{2}$$

At $\theta=\pi/2$, $K=(1/\sqrt{2})^3=\sqrt{2}/4$; at $\theta=4\pi/3$, $K=(\sqrt{3}/2)^3=3\sqrt{3}/8$.

20. From $\mathbf{r}=\rho\mathbf{u}_\rho$, derive formulas for $\mathbf{v}$ and $\mathbf{a}$ in terms of $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$.

Differentiation yields

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{u}_\rho\frac{d\rho}{dt}+\rho\frac{d\mathbf{u}_\rho}{dt}=\mathbf{u}_\rho\frac{d\rho}{dt}+\rho\mathbf{u}_\theta\frac{d\theta}{dt}$$

and

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\mathbf{u}_\rho\frac{d^2\rho}{dt^2}+\mathbf{u}_\theta\frac{d\rho}{dt}\frac{d\theta}{dt}+\rho\mathbf{u}_\theta\frac{d^2\theta}{dt^2}+\mathbf{u}_\theta\frac{d\rho}{dt}\frac{d\theta}{dt}-\rho\mathbf{u}_\rho\left(\frac{d\theta}{dt}\right)^2$$

$$=\mathbf{u}_\rho\left[\frac{d^2\rho}{dt^2}-\rho\left(\frac{d\theta}{dt}\right)^2\right]+\mathbf{u}_\theta\left[\rho\frac{d^2\theta}{dt^2}+2\frac{d\rho}{dt}\frac{d\theta}{dt}\right]$$

21. A particle moves counterclockwise along $\rho=4\sin 2\theta$ with $d\theta/dt=\frac{1}{2}$ rad/sec. (*a*) Express $\mathbf{v}$ and $\mathbf{a}$ in terms of $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$. (*b*) Find $|\mathbf{v}|$ and $|\mathbf{a}|$ when $\theta=\pi/6$.

We have $\quad \mathbf{r}=4\sin 2\theta\,\mathbf{u}_\rho \qquad \dfrac{d\rho}{dt}=8\cos 2\theta\dfrac{d\theta}{dt}=4\cos 2\theta \qquad \dfrac{d^2\rho}{dt^2}=-4\sin 2\theta$

(*a*)
$$\mathbf{v}=\mathbf{u}_\rho\frac{d\rho}{dt}+\rho\mathbf{u}_\theta\frac{d\theta}{dt}=4\mathbf{u}_\rho\cos 2\theta+2\mathbf{u}_\theta\sin 2\theta$$

$$\mathbf{a}=\mathbf{u}_\rho\left[\frac{d^2\rho}{dt^2}-\rho\left(\frac{d\theta}{dt}\right)^2\right]+\mathbf{u}_\theta\left[\rho\frac{d^2\theta}{dt^2}+2\frac{d\rho}{dt}\frac{d\theta}{dt}\right]=-5\mathbf{u}_\rho\sin 2\theta+4\mathbf{u}_\theta\cos 2\theta$$

(*b*) At $\theta=\pi/6$, $\mathbf{u}_\rho=\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$ and $\mathbf{u}_\theta=-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$. Then $\mathbf{v}=\frac{\sqrt{3}}{2}\mathbf{i}+\frac{5}{2}\mathbf{j}$ and $|\mathbf{v}|=\sqrt{7}$; $\mathbf{a}=-\frac{19}{4}\mathbf{i}-\frac{\sqrt{3}}{4}\mathbf{j}$ and $|\mathbf{a}|=\sqrt{91}/2$.

Supplementary Problems

In Problems 22 to 25, find $\tan\psi$ for the given curve at the given points.

22. $\rho=3-\sin\theta$ at $\theta=0$, $\theta=3\pi/4$ *Ans.* -3; $3\sqrt{2}-1$

23. $\rho=a(1-\cos\theta)$ at $\theta=\pi/4$, $\theta=3\pi/2$ *Ans.* $\sqrt{2}-1$; -1

24. $\rho(1-\cos\theta)=a$ at $\theta=\pi/3$, $\theta=5\pi/4$ *Ans.* $-\sqrt{3}/3$; $1+\sqrt{2}$

25. $\rho^2=4\sin 2\theta$ at $\theta=5\pi/12$, $\theta=2\pi/3$ *Ans.* $-1/\sqrt{3}$; $\sqrt{3}$

In Problems 26 to 29, find $\tan\tau$ for the given curve at the given point.

26. $\rho=2+\sin\theta$ at $\theta=\pi/6$ *Ans.* $-3\sqrt{3}$

27. $\rho^2=9\cos 2\theta$ at $\theta=\pi/6$ *Ans.* 0

28. $\rho=\sin^3(\theta/3)$ at $\theta=\pi/2$ *Ans.* $-\sqrt{3}$

29. $2\rho(1-\sin\theta)=3$ at $\theta=\pi/4$ *Ans.* $1+\sqrt{2}$

30. Investigate $\rho = \sin 2\theta$ for horizontal and vertical tangents.

Ans. horizontal tangents at $\theta = 0$, π, 54°44′, 125°16′, 234°44′, 305°16′; vertical tangents at $\theta = \pi/2$, $3\pi/2$, 35°16′, 144°44′, 215°16′, 324°44′

In Problems 31 to 33, find the acute angles of intersection of each pair of curves.

31. $\rho = \sin\theta$, $\rho = \sin 2\theta$ *Ans.* $\phi = 79°6'$ at $\theta = \pi/3$ and $5\pi/3$; $\phi = 0$ at the pole

32. $\rho = \sqrt{2}\sin\theta$, $\rho^2 = \cos 2\theta$ *Ans.* $\phi = \pi/3$ at $\theta = \pi/6$, $5\pi/6$; $\phi = \pi/4$ at the pole

33. $\rho^2 = 16\sin 2\theta$, $\rho^2 = 4\csc 2\theta$ *Ans.* $\phi = \pi/3$ at each intersection

34. Show that each pair of curves intersects at right angles at all points of intersection.
(*a*) $\rho = 4\cos\theta$, $\rho = 4\sin\theta$ (*b*) $\rho = e^{\theta}$, $\rho = e^{-\theta}$
(*c*) $\rho^2\cos 2\theta = 4$, $\rho^2 \sin 2\theta = 9$ (*d*) $\rho = 1 + \cos\theta$, $\rho = 1 - \cos\theta$

35. Find the angle of intersection of the tangents to $\rho = 2 - 4\sin\theta$ at the pole. *Ans.* $2\pi/3$

36. Find the curvature of each of these curves at $P(\rho, \theta)$: (*a*) $\rho = e^{\theta}$; (*b*) $\rho = \sin\theta$; (*c*) $\rho^2 = 4\cos 2\theta$; (*d*) $\rho = 3\sin\theta + 4\cos\theta$.

Ans. (*a*) $1/(\sqrt{2}e^{\theta})$; (*b*) 2; (*c*) $\frac{3}{2}\sqrt{\cos 2\theta}$; (*d*) 2/5

37. Let $\rho = f(\theta)$ be the polar equation of a curve, and let s be the arc length along the curve. Using $x = \rho\cos\theta$, $y = \rho\sin\theta$ and recalling that $\left(\dfrac{ds}{d\theta}\right)^2 = \left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2$, derive $\left(\dfrac{ds}{d\theta}\right)^2 = \rho^2 + (\rho')^2$.

38. Find $ds/d\theta$ for each of the following, assuming s increases in the direction of increasing θ: (*a*) $\rho = a\cos\theta$; (*b*) $\rho = a(1 + \cos\theta)$; (*c*) $\rho = \cos 2\theta$.

Ans. (*a*) a; (*b*) $a\sqrt{2 + 2\cos\theta}$; (*c*) $\sqrt{1 + 3\sin^2 2\theta}$

39. Suppose a particle moves along a curve $\rho = f(\theta)$ with its position at any time t given by $\rho = g(t)$, $\theta = h(t)$.
(*a*) Multiply the relation obtained in Problem 37 by $\left(\dfrac{d\theta}{dt}\right)^2$ to obtain $v^2 = \left(\dfrac{ds}{dt}\right)^2 = \rho^2\left(\dfrac{d\theta}{dt}\right)^2 + \left(\dfrac{d\rho}{dt}\right)^2$.
(*b*) From $\tan\psi = \rho\,\dfrac{d\theta}{d\rho} = \rho\,\dfrac{d\theta/dt}{d\rho/dt}$, obtain $\sin\psi = \dfrac{\rho}{v}\dfrac{d\theta}{dt}$ and $\cos\psi = \dfrac{1}{v}\dfrac{d\rho}{dt}$.

40. Show that $\dfrac{d\mathbf{u}_\rho}{dt} = \mathbf{u}_\theta\,\dfrac{d\theta}{dt}$ and $\dfrac{d\mathbf{u}_\theta}{dt} = -\mathbf{u}_\rho\,\dfrac{d\theta}{dt}$.

41. A particle moves counterclockwise about the cardioid $\rho = 4(1 + \cos\theta)$ with $d\theta/dt = \pi/6$ rad/sec. Express $\mathbf{v}$ and $\mathbf{a}$ in terms of $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$.

Ans. $\mathbf{v} = -\dfrac{2\pi}{3}\mathbf{u}_\rho \sin\theta + \dfrac{2\pi}{3}\mathbf{u}_\theta(1 + \cos\theta)$; $\mathbf{a} = -\dfrac{\pi^2}{9}\mathbf{u}_\rho(1 + 2\cos\theta) - \dfrac{2\pi^2}{9}\mathbf{u}_\theta\sin\theta$

42. A particle moves counterclockwise on $\rho = 8\cos\theta$ with a constant speed of 4 units/sec. Express $\mathbf{v}$ and $\mathbf{a}$ in terms of $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$. *Ans.* $\mathbf{v} = -4\mathbf{u}_\rho\sin\theta + 4\mathbf{u}_\theta\cos\theta$; $\mathbf{a} = -4\mathbf{u}_\rho\cos\theta - 4\mathbf{u}_\theta\sin\theta$

43. If a particle of mass m moves along a path under a force $\mathbf{F}$ which is always directed toward the origin, we have $\mathbf{F} = m\mathbf{a}$ or $\mathbf{a} = \dfrac{1}{m}\mathbf{F}$, so that $a_\theta = 0$. Show that when $a_\theta = 0$, then $\rho^2\dfrac{d\theta}{dt} = k$, a constant, and the radius vector sweeps over area at a constant rate.

44. A particle moves along $\rho = \dfrac{2}{1 - \cos\theta}$ with $a_\theta = 0$. Show that $a_\rho = -\dfrac{k^2}{2}\dfrac{1}{\rho^2}$, where k is defined in Problem 43.

In Problems 45 to 48, find all points of intersection of the given equations.

45. $\rho = 3\cos\theta,\ \rho = 3\sin\theta$ *Ans.* $(0,0),\ (3\sqrt{2}/2,\ \pi/4)$

46. $\rho = \cos\theta,\ \rho = 1 - \cos\theta$ *Ans.* $(0,0),\ (1/2,\ \pi/3),\ (1/2,\ -\pi/3)$

47. $\rho = \theta,\ \rho = \pi$ *Ans.* $(\pi,\ \pi),\ (-\pi,\ -\pi)$

48. $\rho = \sin 2\theta,\ \rho = \cos 2\theta$ *Ans.* $(0,0),\ \left(\dfrac{\sqrt{2}}{2},\ \dfrac{(2n+1)\pi}{6}\right)$ for $n = 0, 1, 2, 3, 4, 5$

Chapter 26

The Law of the Mean

ROLLE'S THEOREM. If $f(x)$ is continuous on the interval $a \le x \le b$, if $f(a) = f(b) = 0$, and if $f'(x)$ exists everywhere on the interval except possibly at the endpoints, then $f'(x) = 0$ for at least one value of x, say $x = x_0$, between a and b.

Geometrically, this means that if a continuous curve intersects the x axis at $x = a$ and $x = b$, and has a tangent at every point between a and b, then there is at least one point $x = x_0$ between a and b where the tangent is parallel to the x axis. (See Fig. 26-1. For a proof, see Problem 11.)

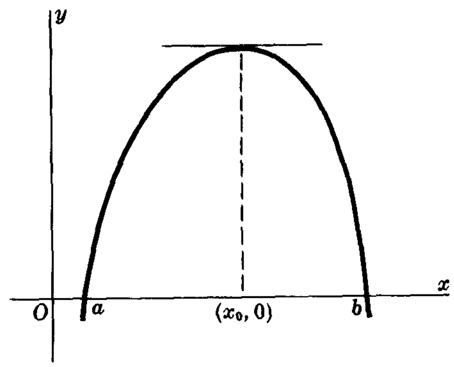


Fig. 26-1

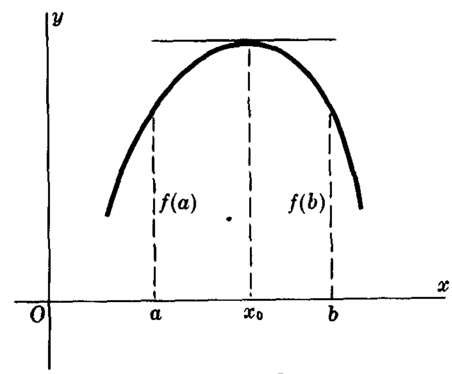


Fig. 26-2

Corollary: If $f(x)$ satisfies the conditions of Rolle's theorem, except that $f(a) = f(b) \neq 0$, then $f'(x) = 0$ for at least one value of x, say $x = x_0$, between a and b.

(See Fig. 26-2 and Problems 1 and 2.)

THE LAW OF THE MEAN. If $f(x)$ is continuous on the interval $a \le x \le b$, and if $f'(x)$ exists everywhere on the interval except possibly at the endpoints, then there is at least one value of x, say $x = x_0$, between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0)$$

Geometrically, this means that if P_1 and P_2 are two points of a continuous curve that has a tangent at each intervening point, then there exists at least one point of the curve between P_1 and P_2 at which the slope of the curve is equal to the slope of P_1P_2. (See Fig. 26-3. For a proof see Problem 12.)

The law of the mean may be put in several useful forms. The first is obtained by multiplication by $b - a$:

$$f(b) = f(a) + (b - a)f'(x_0) \qquad \text{for some } x_0 \text{ between } a \text{ and } b \tag{26.1}$$

A simple change of letter yields

$$f(x) = f(a) + (x - a)f'(x_0) \qquad \text{for some } x_0 \text{ between } a \text{ and } x \tag{26.2}$$

It is clear from Fig. 26-4 that $x_0 = a + \theta(b - a)$ for some θ such that $0 < \theta < 1$. With this replacement, (*26.1*) takes the form

$$f(b) = f(a) + (b - a)f'[a + \theta(b - a)] \qquad \text{for some } \theta \text{ such that } 0 < \theta < 1 \tag{26.3}$$

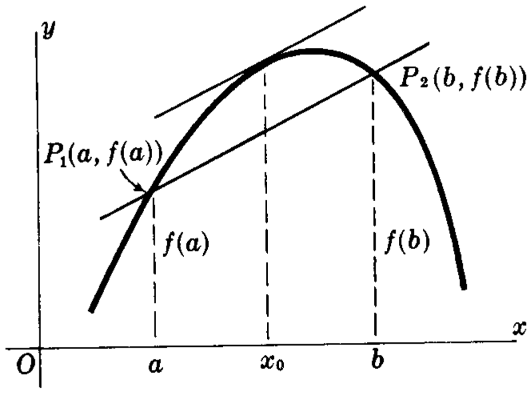


Fig. 26-3

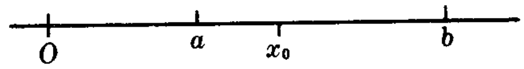


Fig. 26-4

Letting $b - a = h$, we can rewrite (*26.3*) as

$$f(a+h) = f(a) + hf'(a+\theta h) \qquad \text{for some } \theta \text{ such that } 0<\theta<1 \tag{26.4}$$

Finally, if we let $a = x$ and $h = \Delta x$, (*26.4*) becomes

$$f(x+\Delta x) = f(x) + \Delta x\, f'(x+\theta\,\Delta x) \qquad \text{for some } \theta \text{ such that } 0<\theta<1 \tag{26.5}$$

(See Problems 3 to 9.)

GENERALIZED LAW OF THE MEAN. If $f(x)$ and $g(x)$ are continuous on the interval $a \le x \le b$, and if $f'(x)$ and $g'(x)$ exist and $g'(x) \ne 0$ everywhere on the interval except possibly at the endpoints, then there exists at least one value of x, say $x = x_0$, between a and b such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

For the case $g(x) = x$, this becomes the law of the mean. (For a proof, see Problem 13.)

EXTENDED LAW OF THE MEAN. If $f(x)$ and its first $n-1$ derivatives are continuous on the interval $a \le x \le b$, and if $f^{(n)}(x)$ exists everywhere on the interval except possibly at the endpoints, then there is at least one value of x, say $x = x_0$, between a and b such that

$$\begin{aligned} f(b) = f(a) &+ \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots \\ &+ \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(b-a)^n \end{aligned} \tag{26.6}$$

(For a proof, see Problem 15.)

When b is replaced with the variable x, (*26.6*) becomes

$$\begin{aligned} f(x) = f(a) &+ \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &+ \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(x-a)^n \end{aligned} \tag{26.7}$$

for some x_0 between a and x

When a is replaced with 0, (*26.7*) becomes

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(x_0)}{n!}x^n \tag{26.8}$$

for some x_0 between 0 and x

Solved Problems

1. Find the value of x_0 prescribed in Rolle's theorem for $f(x) = x^3 - 12x$ on the interval $0 \le x \le 2\sqrt{3}$.

$f'(x) = 3x^2 - 12 = 0$ when $x = \pm 2$; then $x_0 = 2$ in the prescribed value.

2. Does Rolle's theorem apply to the functions (*a*) $f(x) = \dfrac{x^2 - 4x}{x-2}$ and (*b*) $f(x) = \dfrac{x^2-4x}{x+2}$?

(*a*) $f(x) = 0$ when $x = 0, 4$. Since $f(x)$ is discontinuous at $x = 2$, a point on the interval $0 \le x \le 4$, the theorem does not apply.

(*b*) $f(x) = 0$ when $x = 0, 4$. Here $f(x)$ is discontinuous at $x = -2$, a point not on the interval $0 \le x \le 4$. Moreover, $f'(x) = (x^2 + 4x - 8)/(x+2)^2$ exists everywhere except at $x = -2$. Hence, the theorem applies and $x_0 = 2(\sqrt{3} - 1)$, the positive root of $x^2 + 4x - 8 = 0$.

3. Find the value of x_0 prescribed by the law of the mean, given $f(x) = 3x^2 + 4x - 3$, $a = 1$, $b = 3$.

Using (*26.1*) with $f(a) = f(1) = 4$, $f(b) = f(3) = 36$, $f'(x_0) = 6x_0 + 4$, and $b - a = 2$, we have $36 = 4 + 2(x_0 + 4) = 12x_0 + 12$ and $x_0 = 2$.

4. Use the law of the mean to approximate $\sqrt[6]{65}$.

Let $f(x) = \sqrt[6]{x}$, $a = 64$, and $b = 65$, and apply (*26.1*), obtaining

$$f(65) = f(64) + \frac{65 - 64}{6x_0^{5/6}}, \qquad 64 < x_0 < 65$$

Since x_0 is not known, take $x_0 = 64$; then approximately, $\sqrt[6]{65} = \sqrt[6]{64} + 1/(6\sqrt[6]{64^5}) = 2 + 1/192 = 2.00521$.

5. A circular hole 4 in in diameter and 1 ft deep in a metal block is rebored to increase the diameter to 4.12 in. Estimate the amount of metal removed.

The volume of a circular hole of radius x in and depth 12 in is given by $V = f(x) = 12\pi x^2$. We are to estimate $f(2.06) - f(2)$. By the law of the mean,

$$f(2.06) - f(2) = 0.06f'(x_0) = 0.06(24\pi x_0), \qquad 2 < x_0 < 2.06$$

Take $x_0 = 2$; then, approximately, $f(2.06) - f(2) = 0.06(24\pi)(2) = 2.88\pi \text{ in}^3$.

6. Apply the law of the mean to $y = f(x)$, $a = x$, $b = x + \Delta x$ with all conditions satisfied to show that $\Delta y = f'(x)\,\Delta x$ approximately.

We have $\qquad \Delta y = f(x + \Delta x) - f(x) = (x + \Delta x - x)f'(x_0), \qquad x < x_0 < x + \Delta x$

Take $x_0 = x$; then approximately $\Delta y = f'(x)\,\Delta x$.

7. Use the law of the mean to show $\sin x < x$ for $x > 0$.

Since $\sin x \le 1$, obviously $\sin x < x$ when $x > 1$. For $0 \le x \le 1$, take $f(x) = \sin x$ with $a = 0$ and apply (*26.2*):

$$\sin x = \sin 0 + x \cos x_0 = x \cos x_0, \qquad 0 < x_0 < x$$

Now on this interval $\cos x_0 < 1$ so $x \cos x_0 < x$; hence, $\sin x < x$.

8. Use the law of the mean to show $\dfrac{x}{1+x} < \ln(1+x) < x$ for $-1 < x < 0$ and for $x > 0$.

Apply (*26.4*) with $f(x) = \ln x$, $a = 1$, and $h = x$:

$$\ln(1+x) = \ln 1 + x\,\frac{1}{1+\theta x} = \frac{x}{1+\theta x}, \qquad 0<\theta<1$$

When $x>0$, $1<1+\theta x<1+x$; hence, $1>\frac{1}{1+\theta x}>\frac{1}{1+x}$ and $x>\frac{x}{1+\theta x}>\frac{x}{1+x}$.

When $-1<x<0$, $1>1+\theta x>1+x$; hence, $1<\frac{1}{1+\theta x}<\frac{1}{1+x}$ and $x>\frac{x}{1+\theta x}>\frac{x}{1+x}$.

In each case, $\frac{x}{1+\theta x}<x$ and $\ln(1+x)=\frac{x}{1+\theta x}<x$; also, $\frac{x}{1+\theta x}>\frac{x}{1+x}$ and $\ln(1+x)=\frac{x}{1+\theta x}>\frac{x}{1+x}$. Hence, $\frac{x}{1+x}<\ln(1+x)<x$ when $-1<x<0$ and when $x>0$.

9. Use the law of the mean to show $\sqrt{1+x}<1+\frac{1}{2}x$ for $-1<x<0$ and for $x>0$.

Take $f(x)=\sqrt{x}$ and use (*26.4*) with $a=1$ and $h=x$:

$$\sqrt{1+x} = 1+\frac{x}{2\sqrt{1+\theta x}}, \qquad 0<\theta<1$$

When $x>0$, $\sqrt{1+\theta x}<\sqrt{1+x}$ and $\frac{x}{2\sqrt{1+\theta x}}>\frac{x}{2\sqrt{1+x}}$; when $-1<x<0$, $\sqrt{1+\theta x}>\sqrt{1+x}$ and $\frac{x}{2\sqrt{1+\theta x}}>\frac{x}{2\sqrt{1+x}}$.

In each case, $\sqrt{1+x}=1+\frac{x}{2\sqrt{1+\theta x}}>1+\frac{x}{2\sqrt{1+x}}$. Multiplying the outer inequality by $\sqrt{1+x}>0$, we have $1+x>\sqrt{1+x}+\frac{1}{2}x$ or $\sqrt{1+x}<1+\frac{1}{2}x$.

10. Find a value x_0 as prescribed by the generalized law of the mean, given $f(x)=3x+2$ and $g(x)=x^2+1$, $1\le x\le 4$.

We are to find x_0 so that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f(4)-f(1)}{g(4)-g(1)} = \frac{14-5}{17-2} = \frac{3}{5} = \frac{f'(x_0)}{g'(x_0)} = \frac{3}{2x_0}$$

Then $2x_0=5$ and $x_0=\frac{5}{2}$.

11. Prove Rolle's theorem: If $f(x)$ is continuous on the interval $a\le x\le b$, if $f(a)=f(b)=0$, and if $f'(x)$ exists everywhere on the interval except possibly at the endpoints, then $f'(x)=0$ for at least one value of x, say $x=x_0$, between a and b.

If $f(x)=0$ throughout the interval, then also $f'(x)=0$ and the theorem is proved. Otherwise, if $f(x)$ is positive (negative) somewhere on the interval, it has a relative maximum (minimum) at some $x=x_0$, $a<x_0<b$ (see Property 8.2), and $f'(x_0)=0$.

12. Prove the law of the mean: If $f(x)$ is continuous on the interval $a\le x\le b$, and if $f'(x)$ exists everywhere on the interval except possibly at the endpoints, then there is a value of x, say $x=x_0$, between a and b such that $\frac{f(b)-f(a)}{b-a}=f'(x_0)$.

Refer to Fig. 26-3. The equation of the secant line P_1P_2 is $y=f(b)+K(x-b)$ where $K=\frac{f(b)-f(a)}{b-a}$. At any point x on the interval $a<x<b$, the vertical distance from the secant line to the curve is $F(x)=f(x)-f(b)-K(x-b)$. Now $F(x)$ satisfies the conditions of Rolle's theorem (check this); hence, $F'(x)=f'(x)-K=0$ for some $x=x_0$ between a and b. Thus,

$$K=f'(x_0)=\frac{f(b)-f(a)}{b-a}$$

as was to be proved.

13. Prove the generalized law of the mean: If $f(x)$ and $g(x)$ are continuous on the interval $a \le x \le b$, and if $f'(x)$ and $g'(x)$ exist and $g'(x) \ne 0$ everywhere on the interval except possibly at the endpoints, then there exists at least one value of x, say $x = x_0$, between a and b such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(x_0)}{g'(x_0)}$.

Suppose $g(b) = g(a)$; then by the corollary to Rolle's theorem, $g'(x) = 0$ for some x between a and b. But this is contrary to the hypothesis; thus $g(b) \ne g(a)$.

Now set $\dfrac{f(b)-f(a)}{g(b)-g(a)} = K$, a constant, and form the function $F(x) = f(x) - f(b) - K[g(x) - g(b)]$. This function satisfies the conditions of Rolle's theorem (check this), so that $F'(x) = f'(x) - Kg'(x) = 0$ for at least one value of x, say $x = x_0$, between a and b. Thus,

$$K = \frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

as was to be proved.

14. A curve $y = f(x)$ is *concave upward* on $a < x < b$ if, for any arc PQ of the curve in that interval, the curve lies below the chord PQ; and it is *concave downward* if it lies above all such chords. Prove: If $f(x)$ and $f'(x)$ are continuous on $a \le x \le b$, and if $f'(x)$ has the same sign on $a < x < b$, then

1. $f(x)$ is concave upward on $a < x < b$ when $f''(x) > 0$.
2. $f(x)$ is concave downward on $a < x < b$ when $f''(x) < 0$.

The equation of the chord PQ joining $P(a, f(a))$ and $Q(b, f(b))$ is $y = f(a) + (x-a)\dfrac{f(b)-f(a)}{b-a}$. Let A and B be points on the arc and chord, respectively, having abscissa $x = c$, where $a < c < b$ (Fig. 26-5). The corresponding ordinates are $f(c)$ and

$$f(a) + (c-a)\frac{f(b)-f(a)}{b-a} = \frac{(b-c)f(a) + (c-a)f(b)}{b-a}$$

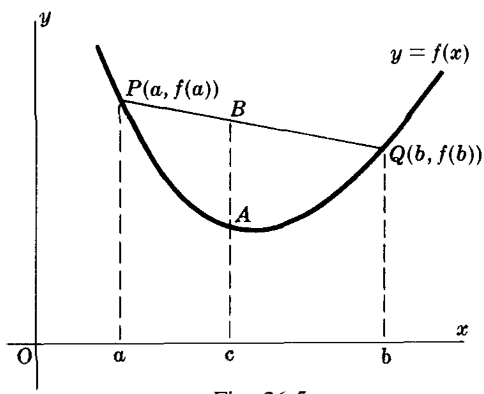


Fig. 26-5

We first must prove $f(c) < \dfrac{(b-c)f(a) + (c-a)f(b)}{b-a}$ when $f''(x) > 0$. By the law of the mean, $\dfrac{f(c)-f(a)}{c-a} = f'(\xi)$, where ξ is between a and c, and $\dfrac{f(b)-f(c)}{b-c} = f'(\eta)$, where η is between c and b. Since $f''(x) > 0$ on $a < x < b$, $f'(x)$ is an increasing function on the interval and $f'(\xi) < f'(\eta)$. Thus $\dfrac{f(c)-f(a)}{c-a} < \dfrac{f(b)-f(c)}{b-c}$, from which it follows that

$$f(c) < \frac{(b-c)f(a) + (c-a)f(b)}{b-a}$$

as required.

The proof of the second part is left as an exercise for the reader.

15. Prove: If $f(x)$ and its first $(n-1)$ derivatives are continuous on the interval $a \le x \le b$, and if $f^{(n)}(x)$ exists everywhere on the interval except possibly at the endpoints, then there is a value of x, say $x = x_0$, between a and b such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(b-a)^n$$

For the case $n = 1$, this becomes the law of the mean. The following proof parallels that of Problem 12. Let K be defined by

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + K(b-a)^n \qquad (1)$$

and consider

$$F(x) = f(x) - f(b) + \frac{f'(x)}{1!}(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} + K(b-x)^n$$

Now $F(a) = 0$ by (*1*), and $F(b) = 0$. By Rolle's theorem, there exists an $x = x_0$, where $a < x_0 < b$, such that

$$\begin{aligned} F'(x_0) &= f'(x_0) + [f''(x_0)(b-x_0) - f'(x_0)] + \left[\frac{f'''(x_0)}{2!}(b-x_0)^2 - f''(x_0)(b-x_0)\right] \\ &\quad + \cdots + \left[\frac{f^{(n)}(x_0)}{(n-1)!}(b-x_0)^{n-1} - \frac{f^{(n-1)}(x_0)}{(n-2)!}(b-x_0)^{n-2}\right] - Kn(b-x_0)^{n-1} \\ &= \frac{f^{(n)}(x_0)}{(n-1)!}(b-x_0)^{n-1} - Kn(b-x_0)^{n-1} = 0 \end{aligned}$$

Then $K = \dfrac{f^{(n)}(x_0)}{n!}$, and (*1*) becomes

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(b-a)^n$$

Supplementary Problems

16. Find a value for x_0 as prescribed by Rolle's theorem, given:

(*a*) $f(x) = x^2 - 4x + 3,\ 1 \le x \le 3$ *Ans.* $x_0 = 2$
(*b*) $f(x) = \sin x,\ 0 \le x \le \pi$ *Ans.* $x_0 = \frac{1}{2}\pi$
(*c*) $f(x) = \cos x,\ \pi/2 < x < 3\pi/2$ *Ans.* $x_0 = \pi$

17. Find a value for x_0 as prescribed by the law of the mean, given:

(*a*) $y = x^3,\ 0 \le x \le 6$ *Ans.* $x_0 = 2\sqrt{3}$
(*b*) $y = ax^2 + bx + c,\ x_1 \le x \le x_2$ *Ans.* $x_0 = \frac{1}{2}(x_1 + x_2)$
(*c*) $y = \ln x,\ 1 \le x \le 2e$ *Ans.* $x_0 = \dfrac{2e-1}{1+\ln 2}$

18. Use the law of the mean to approximate (*a*) $\sqrt{15}$; (*b*) $(3.001)^3$; (*c*) $1/999$.

Ans. (*a*) 3.875, (*b*) 27.027, (*c*) 0.001 001

19. Use the law of the mean to prove (*a*) $\tan x > x,\ 0 < x < \frac{1}{2}\pi$; (*b*) $\dfrac{x}{1+x^2} < \operatorname{Arctan} x < x,\ x > 0$; (*c*) $x < \operatorname{Arcsin} x < \dfrac{x}{\sqrt{1-x^2}},\ 0 < x < 1$.

20. Show that $|f(x) - f(x_1)| \le |x - x_1|$, x_1 being any number, when (*a*) $f(x) = \sin x$; (*b*) $f(x) = \cos x$.

21. Use the law of the mean to prove:

(*a*) If $f'(x) = 0$ everywhere on the interval $a \le x \le b$, then $f(x) = f(a) = c$, a constant, everywhere on the interval.

(*b*) On a given interval $a \le x \le b$, $f(x)$ increases as x increases if $f'(x) > 0$ throughout the interval. (*Hint*: Let $x_1 < x_2$ be two points on the interval; then $f(x_2) = f(x_1) + (x_2 - x_1)f'(x_0)$, $x_1 < x_0 < x_2$.)

22. Use the theorem of Problem 21(*a*) to prove: If $f(x)$ and $g(x)$ are different but $f'(x) = g'(x)$ throughout an interval, then $f(x) - g(x) = c \ne 0$, a constant, on the interval.

23. Define a *bend point* of $f(x)$ to be a critical point $x = x_0$ for which $f'(x)$ changes sign as x increases through $x = x_0$. Let $x_1 < x_2 < \cdots < x_{m-1} < x_m$ be the distinct bend points of $f(x)$. Show that $f(x) = 0$ has at most one real root on each of the intervals $x < x_1$, $x_1 < x < x_2, \ldots, x_{m-1} < x < x_m$, $x > x_m$.

24. Prove: If $f(x)$ is a polynomial of degree n and $f(x) = 0$ has n simple real roots, then $f'(x) = 0$ has exactly $n - 1$ simple real roots.

25. Show that $x^3 + px + q = 0$ has (*a*) one real root if $p > 0$, and (*b*) three real roots if $4p^3 + 27q^2 < 0$.

26. Find a value x_0 as prescribed by the generalized law of the mean, given:

(*a*) $f(x) = x^2 + 2x - 3$, $g(x) = x^2 - 4x + 6$; $a = 0$, $b = 1$ *Ans.* $\frac{1}{2}$

(*b*) $f(x) = \sin x$, $g(x) = \cos x$; $a = \pi/6$, $b = \pi/3$. *Ans.* $\frac{1}{4}\pi$

27. Use (*26.8*) to show:

(*a*) $\sin x$ can be approximated by x with allowable error 0.005 for $x < 0.31$. (*Hint*: For $n = 3$, $\sin x = x - \frac{1}{6}x^3 \cos x_0$. Set $\frac{1}{6}|x^3 \cos x_0| \le \frac{1}{6}|x^3| < 0.005$.)

(*b*) $\sin x$ can be approximated by $x - x^3/6$ with allowable error 0.00005 for $x < 0.359$.

Chapter 27

Indeterminate Forms

THE DERIVATIVE of a differentiable function $f(x)$ is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} \tag{27.1}$$

Since the limit of both the numerator and the denominator of the fraction is zero, it is customary to call *(27.1)* *indeterminate* of the type 0/0. Other examples are found in Problem 6 of Chapter 7.

Similarly, it is customary to call $\lim_{x\to\infty} \frac{3x-2}{9x+7}$ (see Problem 7 of Chapter 7) indeterminate of the type ∞/∞. These symbols 0/0, ∞/∞, and others ($0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞) to be introduced later must not be taken literally; they are merely convenient labels for distinguishing types of behavior at certain limits.

INDETERMINATE TYPE 0/0; L'HOSPITAL'S RULE. If a is a number, if $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$ for all x on some interval $0 < |x-a| < \delta$, and if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then, when $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is infinite,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{(L'Hospital's rule)}$$

EXAMPLE 1: $\lim_{x\to 3} \frac{x^4-81}{x-3}$ is indeterminate of type 0/0. Because

$$\lim_{x\to 3} \frac{\frac{d}{dx}(x^4-81)}{\frac{d}{dx}(x-3)} = \lim_{x\to 3} 4x^3 = 108, \qquad \text{we have} \qquad \lim_{x\to 3} \frac{x^4-81}{x-3} = 108$$

(See Problems 1 to 7.)

Note: L'Hospital's rule remains valid when $\lim_{x\to a}$ is replaced by the one-sided limits $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$.

INDETERMINATE TYPE ∞/∞. The conclusion of l'Hospital's rule is unchanged if one or both of the following changes are made in the hypotheses:

1. "$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$" is replaced by "$\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$."
2. "a is a number" is replaced by "$a = +\infty, -\infty$, or ∞" and "$0 < |x-a| < \delta$" is replaced by "$|x| > M$."

EXAMPLE 2: $\lim_{x\to +\infty} \frac{x^2}{e^x}$ is indeterminate of type ∞/∞. Then l'Hospital's rule gives

$$\lim_{x\to+\infty} \frac{x^2}{e^x} = \lim_{x\to+\infty} \frac{2x}{e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = 0$$

(See Problems 9 to 11.)

INDETERMINATE TYPES $0 \cdot \infty$ and $\infty - \infty$. These may be handled by first transforming to one of the types $0/0$ or ∞/∞. For example:

$$\lim_{x \to +\infty} x^2 e^{-x} \text{ is of type } 0 \cdot \infty \quad \text{but} \quad \lim_{x \to +\infty} \frac{x^2}{e^x} \text{ is of type } \infty/\infty$$

$$\lim_{x \to 0} \left(\csc x - \frac{1}{x}\right) \text{ is of type } \infty - \infty \quad \text{but} \quad \lim_{x \to 0} \left(\frac{x - \sin x}{x \sin x}\right) \text{ is of type } 0/0$$

(See Problems 13 to 16.)

INDETERMINATE TYPES 0^0, ∞^0, and 1^∞. If $\lim y$ is one of these types, then $\lim (\ln y)$ is of the type $0 \cdot \infty$.

EXAMPLE 3: Evaluate $\lim_{x \to 0} (\sec^3 2x)^{\cot^2 3x}$.

This is of the type 1^∞. Let $y = (\sec^3 2x)^{\cot^2 3x}$; then $\ln y = \cot^2 3x \ln \sec^3 2x = \dfrac{3 \ln \sec 2x}{\tan^2 3x}$ and $\lim_{x \to 0} \ln y$ is of the type $0/0$. L'Hospital's rule gives

$$\lim_{x \to 0} \frac{3 \ln \sec 2x}{\tan^2 3x} = \lim_{x \to 0} \frac{6 \tan 2x}{6 \tan 3x \sec^2 3x} = \lim_{x \to 0} \frac{\tan 2x}{\tan 3x}$$

since $\lim_{x \to 0} \sec^2 3x = 1$, and the last limit above is of the type $0/0$. L'Hospital's rule now gives

$$\lim_{x \to 0} \frac{\tan 2x}{\tan 3x} = \lim_{x \to 0} \frac{2 \sec^2 2x}{3 \sec^2 3x} = \frac{2}{3}$$

Since $\lim_{x \to 0} \ln y = \frac{2}{3}$, $\lim_{x \to 0} y = \lim_{x \to 0} (\sec^3 2x)^{\cot^2 3x} = e^{2/3}$.

(See Problems 17 to 19.)

Solved Problems

1. Prove l'Hospital's rule: If a is a number, if $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$ for all x on some interval $0 < |x - a| < \delta$, and if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\text{If } \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists, } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

When b is replaced by x in the generalized law of the mean (Chapter 26), we have, since $f(a) = g(a) = 0$,

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

where x_0 is between a and x. Now $x_0 \to a$ as $x \to a$; hence,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x_0 \to a} \frac{f'(x_0)}{g'(x_0)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

2. Evaluate $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4}$.

When $x \to 2$, both numerator and denominator approach 0. Hence the rule applies, and $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \dfrac{2x + 1}{2x} = \dfrac{5}{4}$.

3. Evaluate $\lim_{x\to 0} \dfrac{x + \sin 2x}{x - \sin 2x}$.

When $x \to 0$, both numerator and denominator approach 0. Hence the rule applies, and $\lim_{x\to 0} \dfrac{x + \sin 2x}{x - \sin 2x} = \lim_{x\to 0} \dfrac{1 + 2\cos 2x}{1 - 2\cos 2x} = \dfrac{1+2}{1-2} = -3$.

4. Evaluate $\lim_{x\to 0} \dfrac{e^x - 1}{x^2}$.

L'Hospital's rule gives $\lim_{x\to 0} \dfrac{e^x - 1}{x^2} = \lim_{x\to 0} \dfrac{e^x}{2x} = \infty$.

5. Evaluate $\lim_{x\to 0} \dfrac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}$.

When $x \to 0$, both numerator and denominator approach 0. Hence the rule applies and

$$\lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2} = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x}$$

Since the resulting function is indeterminate of the type 0/0, we apply the rule to it:

$$\lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2} = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2\cos 2x - 2}$$

Again, the resulting function is indeterminate of the type 0/0. With the understanding that each equality is justified, we obtain, in succession,

$$\lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2} = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2\cos 2x - 2}$$
$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{-4\sin 2x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{-8\cos 2x} = -\frac{1}{4}$$

6. Criticize: $\lim_{x\to 2} \dfrac{x^3 - x^2 - x - 2}{x^3 - 3x^2 + 3x - 2} = \lim_{x\to 2} \dfrac{3x^2 - 2x - 1}{3x^2 - 6x + 3} = \lim_{x\to 2} \dfrac{6x - 2}{6x - 6} = \lim_{x\to 2} \dfrac{6}{6} = 1$.

The given function is indeterminate of the type 0/0, and the rule applies. But the resulting function is not indeterminate (the limit is 7/3); hence, the succeeding applications of the rule are not justified. This is a fairly common error.

7. Criticize: $\lim_{x\to 1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 2x^2 + x} = \dfrac{3x^2 - 2x - 1}{3x^2 - 4x + 1} = \dfrac{6x - 2}{6x - 4} = 2$.

The correct statement is $\lim_{x\to 1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 2x^2 + x} = \lim_{x\to 1} \dfrac{3x^2 - 2x - 1}{3x^2 - 4x + 1} = \lim_{x\to 1} \dfrac{6x - 2}{6x - 4} = 2$. The fact that the limit is correct does not justify the series of incorrect statements in obtaining it.

8. Evaluate $\lim_{x\to \pi^+} \dfrac{\sin x}{\sqrt{x - \pi}}$.

$$\lim_{x\to \pi^+} \frac{\sin x}{\sqrt{x - \pi}} = \lim_{x\to \pi^+} \frac{\cos x}{\frac{1}{2}(x - \pi)^{-1/2}} = \lim_{x\to \pi^+} 2(x - \pi)^{1/2} \cos x = 0$$

Here the approach must be from the right, since otherwise $(x - \pi)^{1/2}$ is imaginary.

9. Evaluate $\lim_{x\to +\infty} \dfrac{\ln x}{x}$.

When $x \to +\infty$, both numerator and denominator approach $+\infty$. Then l'Hospital's rule gives $\lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to +\infty} \frac{1/x}{1} = 0$.

10. Evaluate $\lim_{x \to 0^+} \frac{\ln \sin x}{\ln \tan x}$.

$$\lim_{x \to 0^+} \frac{\ln \sin x}{\ln \tan x} = \lim_{x \to 0^+} \frac{\cos x/\sin x}{\sec^2 x/\tan x} = \lim_{x \to 0^+} \cos^2 x = 1$$

11. Evaluate $\lim_{x \to 0} \frac{\cot x}{\cot 2x}$.

We have $$\lim_{x \to 0} \frac{\cot x}{\cot 2x} = \lim_{x \to 0} \frac{\csc^2 x}{2\csc^2 2x} = \lim_{x \to 0} \frac{\csc^2 x \cot x}{4\csc^2 2x \cot 2x}$$

Here each application of the rule results in an indeterminate form of the type ∞/∞. Instead, we try a trigonometric substitution:

$$\lim_{x \to 0} \frac{\cot x}{\cot 2x} = \lim_{x \to 0} \frac{\tan 2x}{\tan x} = \lim_{x \to 0} \frac{2\sec^2 2x}{\sec^2 x} = 2$$

12. Let $\lim_{x \to +\infty} f(x) = 0$ and $\lim_{x \to +\infty} g(x) = 0$. Prove: If $\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = L$.

Let $x = 1/y$. As $x \to +\infty$, $y \to 0^+$ and $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{y \to 0^+} \frac{f(1/y)}{g(1/y)}$. Then

$$L = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to 0^+} \frac{f'(1/y)}{g'(1/y)} = \lim_{y \to 0^+} \frac{-f'(1/y)y^{-2}}{-g'(1/y)y^{-2}} = \lim_{y \to 0^+} \frac{\frac{d}{dy} f(1/y)}{\frac{d}{dy} g(1/y)}$$

$$= \lim_{y \to 0^+} \frac{f(1/y)}{g(1/y)} = \lim_{x \to +\infty} \frac{f(x)}{g(x)}$$

13. Evaluate $\lim_{x \to 0^+} (x^2 \ln x)$.

As $x \to 0^+$, $x^2 \to 0$ and $\ln x \to -\infty$. Then $\frac{\ln x}{1/x^2}$ has an indeterminate limit of type ∞/∞.

$$\lim_{x \to 0^+} (x^2 \ln x) = \lim_{x \to 0^+} \frac{\ln x}{1/x^2} = \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} \left(-\frac{1}{2}x^2\right) = 0$$

In Problems 14 to 16, evaluate the leftmost limit.

14. $\lim_{x \to \pi/4} (1 - \tan x)\sec 2x = \lim_{x \to \pi/4} \frac{1 - \tan x}{\cos 2x} = \lim_{x \to \pi/4} \frac{-\sec^2 x}{-2\sin 2x} = 1$

15. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x \to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \to 0} \frac{e^x - 1}{xe^x + e^x - 1} = \lim_{x \to 0} \frac{e^x}{xe^x + 2e^x} = \frac{1}{2}$

16. $\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim_{x \to 0} \frac{\sin x}{\cos x} = 0$

17. Evaluate $\lim_{x \to 1} x^{1/(x-1)}$. (This is of the type 1^∞.)

Let $y = x^{1/(x-1)}$. Then $\ln y = \dfrac{\ln x}{x-1}$ has an indeterminate limit of type $\dfrac{0}{0}$. The rule gives

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1$$

Since $\ln y \to 1$ as $x \to 1$, it must be that $y \to e$ as $x \to 1$. Thus the required limit is e.

18. Evaluate $\lim_{x\to \frac{1}{2}\pi^-} (\tan x)^{\cos x}$. (This is of type ∞^0.)

Let $y = (\tan x)^{\cos x}$. Then $\ln y = \cos x \ln \tan x = \dfrac{\ln \tan x}{\sec x}$ has a limit of type $\dfrac{\infty}{\infty}$. The rule gives

$$\lim_{x\to \frac{1}{2}\pi^-} \ln y = \lim_{x\to \frac{1}{2}\pi^-} \frac{\ln \tan x}{\sec x} = \lim_{x\to \frac{1}{2}\pi^-} \frac{\sec^2 x/\tan x}{\sec x \tan x} = \lim_{x\to \frac{1}{2}\pi^-} \frac{\cos x}{\sin^2 x} = 0$$

Since $\ln y \to 0$ as $x \to \frac{1}{2}\pi^-$, $y \to 1$. Thus, the required limit is 1.

19. Evaluate $\lim_{x\to 0^+} x^{\sin x}$. (This is of type 0^0.)

Let $y = x^{\sin x}$. Then $\ln y = \sin x \ln x = \dfrac{\ln x}{\csc x}$ has an indeterminate limit of type $\dfrac{\infty}{\infty}$.

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x \cos x} = \lim_{x\to 0^+} \frac{2 \sin x \cos x}{x \sin x - \cos x} = 0$$

Since $\ln y \to 0$ as $x \to 0^+$, $y \to 1$. Thus, the required limit is 1.

20. Evaluate $\lim_{x\to +\infty} \dfrac{\sqrt{2+x^2}}{x}$.

By repeated application of l'Hospital's rule, $\lim_{x\to +\infty} \dfrac{\sqrt{2+x^2}}{x} = \lim_{x\to +\infty} \dfrac{x}{\sqrt{2+x^2}} = \lim_{x\to +\infty} \dfrac{\sqrt{2+x^2}}{x} \cdots$.

Obviously, the rule is of no help here. However, we have $\lim_{x\to +\infty} \dfrac{\sqrt{2+x^2}}{x} = \lim_{x\to +\infty} \sqrt{\dfrac{2+x^2}{x^2}} = \lim_{x\to +\infty} \sqrt{\dfrac{2}{x^2} + 1} = 1$.

21. The current in a coil containing a resistance R, an inductance L, and a constant electromotive force E at time t is given by $i = \dfrac{E}{R}(1 - e^{-Rt/L})$. Obtain a suitable formula to be used when R is very small.

$$\lim_{R\to 0} i = \lim_{R\to 0} \frac{E(1 - e^{-Rt/L})}{R} = \lim_{R\to 0} E \frac{t}{L} e^{-Rt/L} = \frac{Et}{L}.$$

Supplementary Problems

In Problems 22 to 63, evaluate the limit on the left to obtain the result on the right.

22. $\lim_{x\to 4} \dfrac{x^4 - 256}{x - 4} = 256$

23. $\lim_{x\to 4} \dfrac{x^4 - 256}{x^2 - 16} = 32$

24. $\lim_{x\to 3} \dfrac{x^2 - 3x}{x^2 - 9} = \dfrac{1}{2}$

25. $\lim_{x\to 2} \dfrac{e^x - e^2}{x - 2} = e^2$

26. $\lim_{x \to 0} \frac{xe^x}{1-e^x} = -1$

27. $\lim_{x \to 0} \frac{e^x - 1}{\tan 2x} = \frac{1}{2}$

28. $\lim_{x \to -1} \frac{\ln(2+x)}{x+1} = 1$

29. $\lim_{x \to 0} \frac{\cos x - 1}{\cos 2x - 1} = \frac{1}{4}$

30. $\lim_{x \to 0} \frac{e^{2x} - e^{-2x}}{\sin x} = 4$

31. $\lim_{x \to 0} \frac{8^x - 2^x}{4x} = \frac{1}{2} \ln 2$

32. $\lim_{x \to 0} \frac{2 \arctan x - x}{2x - \arcsin x} = 1$

33. $\lim_{x \to 0} \frac{\ln \sec 2x}{\ln \sec x} = 4$

34. $\lim_{x \to 0} \frac{\ln \cos x}{x^2} = -\frac{1}{2}$

35. $\lim_{x \to 0} \frac{\cos 2x - \cos x}{\sin^2 x} = -\frac{3}{2}$

36. $\lim_{x \to +\infty} \frac{\ln x}{\sqrt{x}} = 0$

37. $\lim_{x \to \frac{1}{2}\pi} \frac{\csc 6x}{\csc 2x} = \frac{1}{3}$

38. $\lim_{x \to +\infty} \frac{5x + 2 \ln x}{x + 3 \ln x} = 5$

39. $\lim_{x \to +\infty} \frac{x^4 + x^2}{e^x + 1} = 0$

40. $\lim_{x \to 0^+} \frac{\ln \cot x}{e^{\csc^2 x}} = 0$

41. $\lim_{x \to +\infty} \frac{e^x + 3x^3}{4e^x + 2x^2} = \frac{1}{4}$

42. $\lim_{x \to 0} (e^x - 1) \cos x = 1$

43. $\lim_{x \to -\infty} x^2 e^x = 0$

44. $\lim_{x \to 0} x \csc x = 1$

45. $\lim_{x \to 1} \csc \pi x \ln x = -1/\pi$

46. $\lim_{x \to \frac{1}{2}\pi^-} e^{-\tan x} \sec^2 x = 0$

47. $\lim_{x \to 0} (x - \arcsin x) \csc^3 x = -\frac{1}{6}$

48. $\lim_{x \to 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x-2} \right) = -\frac{1}{4}$

49. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = 0$

50. $\lim_{x \to \frac{1}{2}\pi} (\sec^3 x - \tan^3 x) = \infty$

51. $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{x}{x-1} \right) = -\frac{1}{2}$

52. $\lim_{x \to 0} \left(\frac{4}{x^2} - \frac{2}{1 - \cos x} \right) = -\frac{1}{3}$

53. $\lim_{x \to +\infty} \left(\frac{\ln x}{x} - \frac{1}{\sqrt{x}} \right) = 0$

54. $\lim_{x \to 0^+} x^x = 1$

55. $\lim_{x \to 0} (\cos x)^{1/x} = 1$

56. $\lim_{x \to 0} (e^x + 3x)^{1/x} = e^4$

57. $\lim_{x \to +\infty} (1 - e^{-x})^{e^x} = 1/e$

58. $\lim_{x \to \frac{1}{2}\pi} (\sin x - \cos x)^{\tan x} = 1/e$

59. $\lim_{x \to \frac{1}{2}\pi^-} (\tan x)^{\cos x} = 1$

60. $\lim_{x \to 1} x^{\tan \frac{1}{2}\pi x} = e^{-2/\pi}$

61. $\lim_{x \to +\infty} (1 + 1/x)^x = e$

62. (*a*) $\lim_{x \to 0} \frac{e^x(1 - e^x)}{(1+x)\ln(1-x)} = \lim_{x \to 0} \frac{e^x}{1+x} \lim_{x \to 0} \frac{1 - e^x}{\ln(1-x)} = 1$; (*b*) $\lim_{x \to +\infty} \frac{2^x}{3^{x^2}} = 0$; (*c*) $\lim_{x \to 0^+} \frac{e^{-3/x}}{x^2} = 0$

63. (*a*) $\lim_{x \to +\infty} \frac{\ln^5 x}{x^2} = 0$; (*b*) $\lim_{x \to +\infty} \frac{\ln^{1000} x}{x^5} = 0$

Chapter 28

Differentials

DIFFERENTIALS. For the function $y = f(x)$, we define the following:

1. dx, called the *differential of* x, given by the relation $dx = \Delta x$
2. dy, called the *differential of* y, given by the relation $dy = f'(x)\,dx$

The differential of the independent variable is, by definition, equal to the increment of the variable. But the differential of the dependent variable is *not* equal to the increment of that variable. See Fig. 28-1.

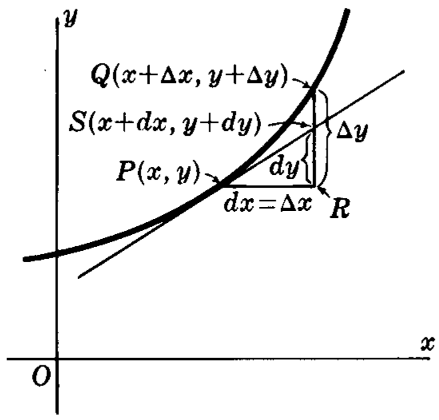


Fig. 28-1

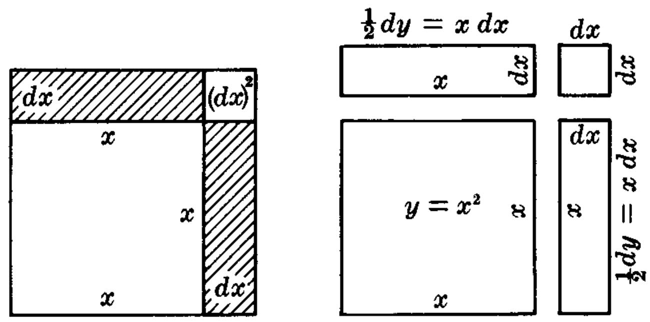


Fig. 28-2

EXAMPLE 1: When $y = x^2$, $dy = 2x\,dx$ while $\Delta y = (x + \Delta x)^2 - x^2 = 2x\,\Delta x + (\Delta x)^2 = 2x\,dx + (dx)^2$. A geometric interpretation is given in Fig. 28-2, where you can see that Δy and dy differ by the small square of area $(dx)^2$.

THE DIFFERENTIAL dy may be found by using the definition $dy = f'(x)\,dx$ or by means of rules obtained readily from the rules for finding derivatives. Some of these are:

$$d(c) = 0 \qquad d(cu) = c\,du \qquad d(uv) = u\,dv + v\,du$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \qquad d(\sin u) = \cos u\,du \qquad d(\ln u) = \frac{du}{u}$$

EXAMPLE 2: Find dy for each of the following:

(*a*) $y = x^3 + 4x^2 - 5x + 6$

$$dy = d(x^3) + d(4x^2) - d(5x) + d(6) = (3x^2 + 8x - 5)\,dx$$

(*b*) $y = (2x^3 + 5)^{3/2}$

$$dy = \tfrac{3}{2}(2x^3 + 5)^{1/2}\,d(2x^3 + 5) = \tfrac{3}{2}(2x^3 + 5)^{1/2}(6x^2\,dx) = 9x^2(2x^3 + 5)^{1/2}\,dx$$

(See Problems 1 to 5.)

APPROXIMATIONS BY DIFFERENTIALS. If $dx = \Delta x$ is relatively small when compared with x, dy is a fairly good approximation of Δy.

EXAMPLE 3: Take $y = x^2 + x + 1$, and let x change from $x = 2$ to $x = 2.01$. The actual change in y is $\Delta y = [(2.01)^2 + 2.01 + 1] - (2^2 + 2 + 1) = 0.0501$. The approximate change in y, obtained by taking $x = 2$ and $dx = 0.01$, is $dy = f'(x)\,dx = (2x + 1)\,dx = [2(2) + 1]0.01 = 0.05$

(See Problems 6 to 10.)

APPROXIMATIONS OF ROOTS OF EQUATIONS. Let $x = x_1$ be a fairly close approximation of a root r of the equation $y = f(x) = 0$, and let $f(x_1) = y_1 \neq 0$. Then y_1 differs from 0 by a small amount. Now if x_1 were changed to r, the corresponding change in $f(x_1)$ would be $\Delta y_1 = -y_1$. An approximation of this change in x_1 is given by $f'(x_1)\,dx_1 = -y_1$ or $dx_1 = -\dfrac{y_1}{f'(x_1)}$. Thus, a second and better approximation of the root r is

$$x_2 = x_1 + dx_1 = x_1 - \frac{y_1}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

A third approximation is $x_3 = x_2 + dx_2 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$, and so on.

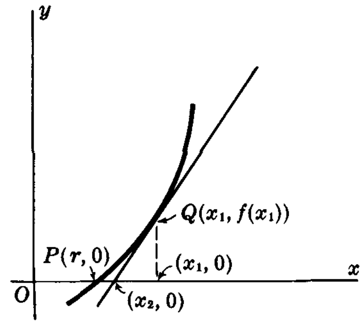


Fig. 28-3

When x_1 is not a sufficiently close approximation of a root, it will be found that x_2 differs materially from x_1. While at times the process of finding these approximations is self-correcting, it is often simpler to make a new first approximation. (See Problems 11 and 12.)

Solved Problems

1. Find dy for each of the following:

(a) $y = \dfrac{x^3 + 2x + 1}{x^2 + 3}$:

$$dy = \frac{(x^2+3)\,d(x^3+2x+1) - (x^3+2x+1)\,d(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)(3x^2+2)\,dx - (x^3+2x+1)(2x)\,dx}{(x^2+3)^2} = \frac{x^4+7x^2-2x+6}{(x^2+3)^2}\,dx$$

(b) $y = \cos^2 2x + \sin 3x$:

$$dy = 2\cos 2x\,d(\cos 2x) + d(\sin 3x) = (2\cos 2x)(-2\sin 2x\,dx) + 3\cos 3x\,dx$$
$$= -4\sin 2x\cos 2x\,dx + 3\cos 3x\,dx = (-2\sin 4x + 3\cos 3x)\,dx$$

(c) $y = e^{3x} + \arcsin 2x$: $\quad dy = \left(3e^{3x} + \dfrac{2}{\sqrt{1-4x^2}}\right) dx$

In Problems 2 to 5, use differentials to obtain dy/dx.

2. $xy + x - 2y = 5$

We have $\quad d(xy) + d(x) - d(2y) = d(5) \quad$ or $\quad x\,dy + y\,dx + dx - 2\,dy = 0$

Then $\quad (x-2)\,dy + (y+1)\,dx = 0 \quad$ and $\quad \dfrac{dy}{dx} = -\dfrac{y+1}{x-2}$

3. $x^3y^2 - 2x^2y + 3xy^2 - 8xy = 6$

Here $\quad 2x^3y\,dy + 3x^2y^2\,dx - 2x^2\,dy - 4xy\,dx + 6xy\,dy + 3y^2\,dx - 8x\,dy - 8y\,dx = 0$

so
$$\frac{dy}{dx} = \frac{8y - 3y^2 + 4xy - 3x^2y^2}{2x^3y - 2x^2 + 6xy - 8x}$$

4. $\dfrac{2x}{y} - \dfrac{3y}{x} = 8$

Here $\quad 2\left(\dfrac{y\,dx - x\,dy}{y^2}\right) - 3\left(\dfrac{x\,dy - y\,dx}{x^2}\right) = 0 \quad$ and $\quad \dfrac{dy}{dx} = \dfrac{2x^2y + 3y^3}{3xy^2 + 2x^3}$

5. $x = 3\cos\theta - \cos 3\theta,\ y = 3\sin\theta - \sin 3\theta$

$$dx = (-3\sin\theta + 3\sin 3\theta)\,d\theta \qquad dy = (3\cos\theta - 3\cos 3\theta)\,d\theta \qquad \frac{dy}{dx} = \frac{\cos\theta - \cos 3\theta}{-\sin\theta + \sin 3\theta}$$

6. Use differentials to approximate (a) $\sqrt[3]{124}$, (b) $\sin 60°1'$.

(a) For $y = x^{1/3}$, $dy = \dfrac{1}{3x^{2/3}}\,dx$. Take $x = 125 = 5^3$ and $dx = -1$. Then $dy = \dfrac{1}{3(125)^{2/3}}(-1) = \dfrac{-1}{75} = -0.0133$ and, approximately, $\sqrt[3]{124} = y + dy = 5 - 0.0133 = 4.9867$.

(b) For $x = 60°$ and $dx = 1' = 0.0003$ rad, $y = \sin x = \sqrt{3}/2 = 0.86603$ and $dy = \cos x\,dx = \frac{1}{2}(0.0003) = 0.00015$. Then, approximately, $\sin 60°1' = y + dy = 0.86603 + 0.00015 = 0.86618$.

7. Compute Δy, dy, and $\Delta y - dy$, given $y = \frac{1}{2}x^2 + 3x$, $x = 2$, and $dx = 0.5$.

$$\Delta y = [\tfrac{1}{2}(2.5)^2 + 3(2.5)] - [\tfrac{1}{2}(2)^2 + 3(2)] = 2.625$$
$$dy = (x+3)\,dx = (2+3)(0.5) = 2.5$$
$$\Delta y - dy = 2.625 - 2.5 = 0.125$$

8. Find the approximate change in the volume V of a cube of side x in caused by increasing the sides by 1%.

$V = x^3$ and $dV = 3x^2\,dx$. When $dx = 0.01x$, $dV = 3x^2(0.01x) = 0.03x^3$ in^3.

9. Find the approximate weight of an 8-ft length of copper tubing if the inside diameter is 1 in and the thickness is 1/8 in. The specific weight of copper is 550 lb/ft^3.

First find the change in volume when the radius $r = \frac{1}{24}$ ft is changed by $dr = \frac{1}{96}$ ft:

$$V = 8\pi r^2 \qquad dV = 16\pi r\,dr = 16\pi\,\frac{1}{24}\,\frac{1}{96} = \frac{\pi}{144}\ \text{ft}^3$$

This is the volume of copper. Its weight is $550(\pi/144) = 12$ lb.

10. For what values of x may $\sqrt[5]{x}$ be used in place of $\sqrt[5]{x+1}$, if the error must be less than 0.001?

When $y = x^{1/5}$ and $dx = 1$, $dy = \frac{1}{5}x^{-4/5}\,dx = \frac{1}{5}x^{-4/5}$.
If $\frac{1}{5}x^{-4/5} < 10^{-3}$, then $x^{-4/5} < 5(10^{-3})$ and $x^{-4} < 5^5(10^{-15})$.
If $x^{-4} < 10(5^5)(10^{-16})$, then $x^4 > \dfrac{10^{16}}{31\,250}$ and $x > \dfrac{10^4}{\sqrt[4]{31\,250}} = 752.1$.

11. Approximate the (real) roots of $x^3 + 2x - 5 = 0$ or $x^3 = 5 - 2x$.

On the same axes, construct the graphs of $y = x^3$ and $y = 5 - 2x$. The abscissas of the points of intersection of the curves are the roots of the given equation. From the graph, it may be seen that there is one root whose approximate value is $x_1 = 1.3$.

A second approximation of this root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3 - \frac{(1.3)^3 + 2(1.3) - 5}{3(1.3)^2 + 2} = 1.3 - \frac{-0.203}{7.07} = 1.3 + 0.03 = 1.33$$

The division above is carried out to yield two decimal places, since there is one zero immediately following the decimal point. This is in accord with a theorem: If in a division, k zeros immediately follow the decimal point in the quotient, the division can be carried out to yield $2k$ decimal places.

A third and fourth approximation are

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.33 - \frac{(1.33)^3 + 2(1.33) - 5}{3(1.33)^2 + 2} = 1.33 - 0.0017 = 1.3283$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.3283 - 0.000\,031\,14 = 1.328\,268\,86$$

12. Approximate the roots of $2\cos x - x^2 = 0$.

The curves $y = 2\cos x$ and $y = x^2$ intersect in two points whose abscissas are approximately 1 and -1. (Note that if r is one root, then $-r$ is the other.)

Using $x_1 = 1$ yields $x_2 = 1 - \dfrac{2\cos 1 - 1}{-2\sin 1 - 2} = 1 + \dfrac{2(0.5403) - 1}{2(0.8415) + 2} = 1 + 0.02 = 1.02.$

Then $x^3 = 1.02 - \dfrac{2\cos(1.02) - (1.02)^2}{-2\sin(1.02) - 2(1.02)} = 1.02 + \dfrac{0.0064}{3.7442} = 1.02 + 0.0017 = 1.0217$. Thus, to four decimal places, the roots are 1.0217 and -1.0217.

Supplementary Problems

13. Find dy for each of the following:

(a) $y = (5 - x)^3$ *Ans.* $-3(5 - x)^2\,dx$

(b) $y = e^{4x^2}$ *Ans.* $8xe^{4x^2}\,dx$

(c) $y = (\sin x)/x$ *Ans.* $\dfrac{x\cos x - \sin x}{x^2}\,dx$

(d) $y = \cos bx^2$ *Ans.* $-2bx\sin bx^2\,dx$

(e) $y = \arccos 2x$ *Ans.* $\dfrac{-2}{\sqrt{1 - 4x^2}}\,dx$

(f) $y = \ln\tan x$ *Ans.* $\dfrac{2\,dx}{\sin 2x}$

14. Find dy/dx as in Problems 2 to 5:

(a) $2xy^3 + 3x^2y = 1$ *Ans.* $-\dfrac{2y(y^2 + 3x)}{3x(2y^2 + x)}$

(b) $xy = \sin(x - y)$ *Ans.* $\dfrac{\cos(x - y) - y}{\cos(x - y) + x}$

(c) $\arctan\dfrac{y}{x} = \ln(x^2 + y^2)$ *Ans.* $\dfrac{2x + y}{x - 2y}$

(d) $x^2\ln y + y^2\ln x = 2$ *Ans.* $-\dfrac{(2x^2\ln y + y^2)y}{(2y^2\ln x + x^2)x}$

15. Use differentials to approximate (*a*) $\sqrt[4]{17}$, (*b*) $\sqrt[5]{1020}$, (*c*) $\cos 59°$, and (*d*) $\tan 44°$.

Ans. (*a*) 2.03125; (*b*) 3.99688; (*c*) 0.5151; (*d*) 0.9651

16. Use differentials to approximate the change in (*a*) x^3 as x changes from 5 to 5.01; (*b*) $1/x$ as x changes from 1 to 0.98. *Ans.* (*a*) 0.75; (*b*) 0.02

17. A circular plate expands under the influence of heat so that its radius increases from 5 in to 5.06 in. Find the approximate increase in area. *Ans.* $0.6\pi = 1.88 \text{ in}^2$

18. A sphere of ice of radius 10 in shrinks to radius 9.8 in. Approximate the decrease in (*a*) volume and (*b*) surface area. *Ans.* (*a*) $80\pi \text{ in}^3$; (*b*) $16\pi \text{ in}^2$

19. The velocity (v ft/sec) attained by a body falling freely a distance h ft from rest is given by $v = \sqrt{64.4h}$. Find the error in v due to an error of 0.5 ft when h is measured as 100 ft. *Ans.* 0.2 ft/sec

20. If an aviator flies around the world at a distance 2 mi above the equator, how many more miles will he travel than a person who travels along the equator? *Ans.* 12.6 mi

21. The radius of a circle is to be measured and its area computed. If the radius can be measured to 0.001 in and the area must be accurate to 0.1 in^2, find the maximum radius for which this process can be used. *Ans.* approximately 16 in

22. If $pV = 20$ and p is measured as 5 ± 0.02, find V. *Ans.* $V = 4 \mp 0.016$

23. If $F = 1/r^2$ and F is measured as 4 ± 0.05, find r. *Ans.* 0.5 ∓ 0.003

24. Find the change in the total surface of a right circular cone when (*a*) the radius remains constant while the altitude changes by a small amount; (*b*) the altitude remains constant while the radius changes by a small amount. *Ans.* (*a*) $\pi rh\, dh/\sqrt{r^2 + h^2}$; (*b*) $\pi\left[\dfrac{h^2 + 2r^2}{\sqrt{r^2 + h^2}} + 2r\right] dr$

25. Find, to four decimal places, (*a*) the real root of $x^3 + 3x + 1 = 0$; (*b*) the smallest root of $e^{-x} = \sin x$; (*c*) the root of $x^2 + \ln x = 2$; (*d*) the root of $x - \cos x = 0$.

Ans. (*a*) -0.3222; (*b*) 0.5885; (*c*) 1.3141; (*d*) 0.7391

Chapter 29

Curve Tracing

SYMMETRY. A curve is symmetric with respect to

1. The x axis, if its equation is unchanged when y is replaced by $-y$
2. The y axis, if its equation is unchanged when x is replaced by $-x$
3. The origin, if its equation is unchanged when x is replaced by $-x$ and y by $-y$ simultaneously.
4. The line $y = x$, if its equation is unchanged when x and y are interchanged

INTERCEPTS. The x intercepts are obtained by setting $y = 0$ in the equation for the curve and solving for x. The y intercepts are obtained by setting $x = 0$ and solving for y.

EXTENT. The *horizontal extent* of a curve is given by the range of x, for example, the intervals of x for which the curve exists. The *vertical extent* is given by the range of y.

A point (x_0, y_0) is called an *isolated point* of a curve if its coordinates satisfy the equation of the curve while those of no other nearby point do.

ASYMPTOTES. An *asymptote* of a curve is a line that comes arbitrarily close to the curve as the curve recedes indefinitely away from the origin (that is, as the abscissa or ordinate of the curve approaches infinity).

The maximum and minimum points, points of inflection, and concavity of a curve are discussed in Chapter 13.

Solved Problems

1. Discuss and sketch the curve $y^2(1+x) = x^2(1-x)$. (See Fig. 29-1.)

We may write the equation of the curve as $y^2 = \dfrac{x^2(1-x)}{1+x}$.

Symmetry: The curve is symmetric with respect to the x axis.

Intercepts: The x intercepts are $x = 0$ and $x = 1$. The y intercept is $y = 0$.

Extent: For $x = 1$, $y = 0$. For $x = -1$, there is no point on the curve. For other values of x, y^2 must be positive so $1 + x$ and $1 - x$ must have the same sign; hence, for points on the curve, x is restricted to $-1 < x < 1$. Thus, $-1 < x \le 1$.

Asymptotes: $y^2 = \dfrac{x^2(1-x)}{1+x}$. Hence, $y \to \infty$ as $x \to -1$. Thus, $x = -1$ is a vertical asymptote.

Maximum and minimum points, etc.: The curve consists of two branches $y = \dfrac{x\sqrt{1-x}}{\sqrt{1+x}}$ and $y = -\dfrac{x\sqrt{1-x}}{\sqrt{1+x}}$. For the first of these,

$$\frac{dy}{dx} = \frac{1 - x - x^2}{(1+x)^{3/2}(1-x)^{1/2}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{x-2}{(1+x)^{5/2}(1-x)^{3/2}}$$

The critical values are $x = 1$ and $(-1 + \sqrt{5})/2$. The point $\left(\dfrac{-1+\sqrt{5}}{2}, \dfrac{(-1+\sqrt{5})\sqrt{\sqrt{5}-2}}{2}\right)$ is a

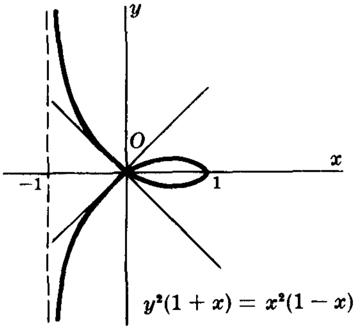


Fig. 29-1

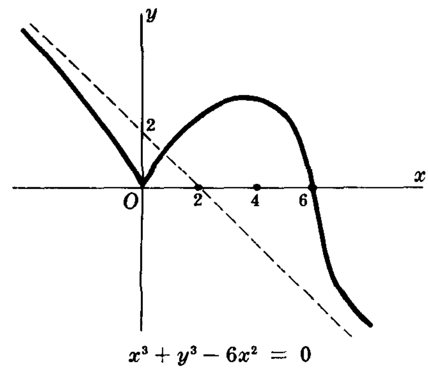


Fig. 29-2

maximum point. There is no point of inflection. The branch is concave downward. By symmetry, there is a minimum point at $\left(\frac{-1+\sqrt{5}}{2}, \frac{(-1+\sqrt{5})\sqrt{\sqrt{5}-2}}{2}\right)$, and the second branch is concave upward.

The curve passes through the origin twice. The tangent lines at the origin are the lines $y = x$ and $y = -x$.

2. Discuss and sketch the curve $y^3 - x^2(6-x) = 0$. (See Fig. 29-2.)

We may write the equation of the curve as $y^3 = x^2(6-x) = 6x^2 - x^3$.

Symmetry: There is no symmetry.

Intercepts: The x intercepts are $x = 0$ and $x = 6$. The y intercept is $y = 0$. y is negative when and only when $x > 6$.

Extent: The curve is defined for all x. As $x \to +\infty$, $y \to -\infty$; as $x \to -\infty$, $y \to +\infty$. Hence, there is no horizontal asymptote.

Maximum and minimum points, etc.: We have $\frac{dy}{dx} = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$ and $\frac{d^2y}{dx^2} = \frac{-8}{x^{4/3}(6-x)^{5/3}}$. The critical values are $x = 0$, $x = 4$, and $x = 6$. When $x = 0$, $y = 0$. Since $y > 0$ to the left and right of the origin, $(0, 0)$ yields a relative minimum.

The point $(4, 2\sqrt[3]{4})$ is a relative maximum point by the second-derivative text. The point $(6, 0)$ is a point of inflection, the curve being concave downward to the the left of $(6, 0)$ and concave upward to the right.

Asymptotes: There are no horizontal or vertical asymptotes. There is an oblique asymptote $y = mx + b$. To find m and b, we expand $(mx+b)^3$ to obtain $m^3x^3 + 3m^2bx^2 + 3mb^2x + b^3$ and set the two leading coefficients, m^3 and $3m^2b$, equal to the corresponding coefficients of $-x^3 + 6x^2$. This gives $m^3 = -1$ and $3m^2b = 6$. Hence, $m = -1$ and $b = 2$, and the asymptote (on the right and left) is the line $y = -x + 2$.

3. Discuss and sketch the curve $y^2(x-1) - x^3 = 0$. (See Fig. 29-3.)

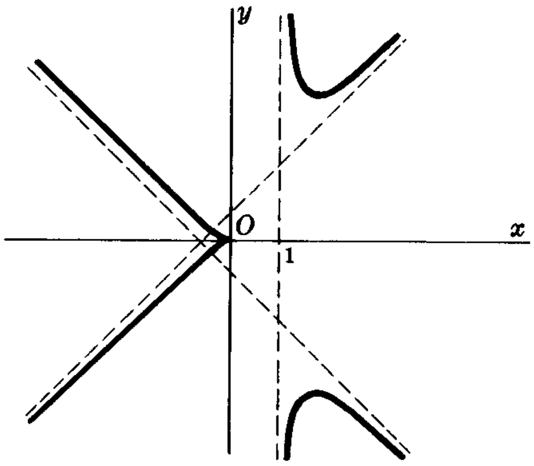
We may write the equation as $y^2 = \frac{x^3}{x-1}$.

Extent: Clearly, the origin is on the graph. At other points, the left side y^2 must be positive, and therefore x^3 and $x - 1$ must have the same sign. Hence, $x > 1$ or $x \le 0$.

Symmetry: The curve is symmetric with respect to the x axis.

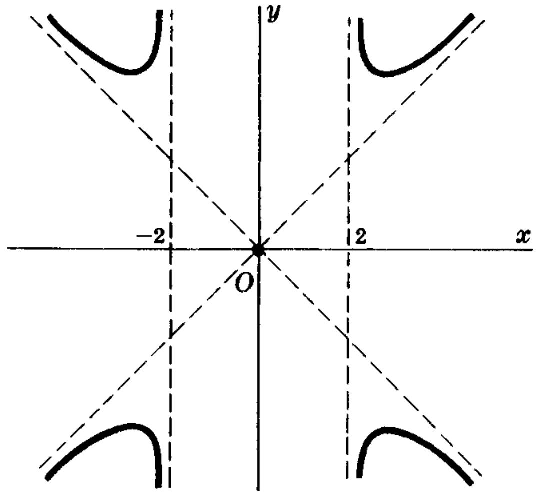
Intercepts: The only intercepts are $x = 0$ and $y = 0$.

Maximum and minimum points, etc.: For the branch $y = x\sqrt{\frac{x}{x-1}}$, we have $\frac{dy}{dx} = \frac{1}{2}(2x-3)\cdot\left[\frac{x}{(x-1)^3}\right]^{1/2}$ and $\frac{d^2y}{dx^2} = \frac{3}{4[x(x-1)^5]^{1/2}}$. The critical values are $x = 0$ and $3/2$. The point $(3/2, 3\sqrt{3}/2)$ is a minimum point. There is no point of inflection. The branch is concave upward. By symmetry, there is a maximum point $(3/2, -3\sqrt{3}/2)$ on the branch $y = -x\sqrt{\frac{x}{x-1}}$, and that branch is concave downward.



$y^2(x-1)-x^3=0$

Fig. 29-3



$y^2(x^2-4)=x^4$

Fig. 29-4

Asymptotes: There is a vertical asymptote $x=1$. Since $y\to\infty$ as $x\to\infty$, there is no horizontal asymptote. To find oblique asymptotes $y=mx+b$, we set $(mx+b)^2=\dfrac{x^3}{x-1}$, obtaining

$$(m^2-1)x^3+(2mb-m^2)x^2+(b^2-2mb)x-b^2=0$$

Setting $m^2-1=0$ and $2mb-m^2=0$, we obtain $m=\pm1$, $b=\pm1/2m$. Thus, the asymptotes are $y=x+\frac{1}{2}$ and $y=-x-\frac{1}{2}$.

4. Discuss and sketch the curve $y^2(x^2-4)=x^4$. (See Fig. 29-4.)

Symmetry: The curve is symmetric with respect to the coordinate axes and the origin.
Intercepts: The intercepts are $x=0$ and $y=0$.
Extent: The curve exists for $x^2>4$, that is, for $x>2$ or $x<-2$, plus the isolated point $(0,0)$.
Maximum and minimum points, etc.: For the portion $y=\dfrac{x^2}{\sqrt{x^2-4}}$, $x>2$, we have $\dfrac{dy}{dx}=\dfrac{x^3-8x}{(x^2-4)^{3/2}}$ and $\dfrac{d^2y}{dx^2}=\dfrac{4x^2+32}{(x^2-4)^{5/2}}$. The critical value is $x=2\sqrt{2}$. The portion is concave upward, and $(2\sqrt{2},4)$ is a relative minimum point. By symmetry, there is a relative minimum point at $(-2\sqrt{2},4)$, and relative maximum points at $(2\sqrt{2},-4)$ and $(-2\sqrt{2},-4)$.

Asymptotes: The lines $x=2$ and $x=-2$ are vertical asymptotes. For the oblique asymptotes, we replace y with $mx+b$ to obtain

$$(m^2-1)x^4+2mbx^3+(b^2-4m^2)x^2-8mbx-4b^2=0$$

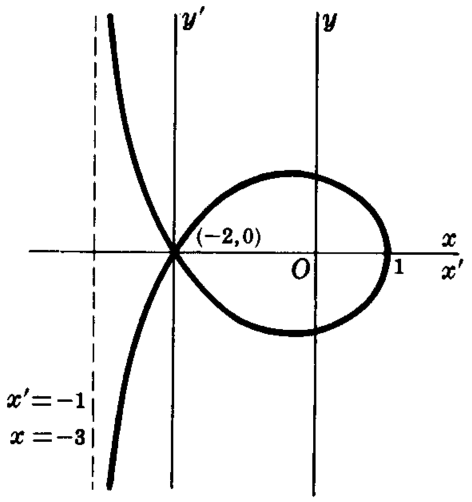
Solving simultaneously $m^2-1=0$ and $mb=0$, we obtain $m=1$, $b=0$ and $m=-1$, $b=0$. The equations of the oblique asymptotes are thus $y=x$ and $y=-x$. They intersect the curve at the origin.

5. Discuss and sketch the curve $(x+3)(x^2+y^2)=4$. (See Fig. 29-5.)

$\dfrac{dy}{dx}=-\dfrac{(x+2)(x+2+\sqrt{3})(x+2-\sqrt{3})}{(x+3)^2y}$. When $x=-2$, $y=0$ and $\dfrac{dy}{dx}$ has the indeterminate form $\dfrac{0}{0}$. But if we let $x=X-2$ and $y=Y$, the equation becomes $Y^2(X+1)+X^3-3X^2=0$.

Symmetry: The curve is symmetric with respect to the x axis.
Intercepts: The intercepts are $X=0$, $X=3$, and $Y=0$.
Extent: The curve is defined on the interval $-1<X\leqq3$ and for all values of Y.
Maximum and minimum points, etc.: For the branch $Y=\dfrac{X\sqrt{3-X}}{\sqrt{X+1}}$,

$$\frac{dY}{dX}=\frac{3-X^2}{(3-X)^{1/2}(X+1)^{3/2}} \quad\text{and}\quad \frac{d^2Y}{dX^2}=\frac{-12}{(3-X)^{3/2}(X+1)^{5/2}}$$



$(x+3)(x^2+y^2) = 4$

Fig. 29-5

The critical values are $X=\sqrt{3}$ and 3. The point $(\sqrt{3}, \sqrt{6\sqrt{3}-9})$ is a maximum point. The branch is concave downward.

By symmetry, $(\sqrt{3}, \sqrt{6\sqrt{3}-9})$ is a minimum point on the other branch, which is concave upward.

Asymptotes: The line $X=-1$ is a vertical asymptote. For the oblique asymptotes, replace Y with $mX+b$ to obtain $(m^2+1)X^3+\cdots=0$. There are no oblique asymptotes. Why?

In the original coordinates, $(\sqrt{3}-2, \sqrt{6\sqrt{3}-9})$ is a maximum point and $(\sqrt{3}-2, -\sqrt{6\sqrt{3}-9})$ is a minimum point. The line $x=-3$ is a vertical asymptote.

6. Discuss and sketch the curve $y = \dfrac{\ln x}{x}$. (See Fig. 29-6.)

Symmetry: There is no symmetry.

Intercepts: The only intercept is $x=1$.

Extent: the curve is defined for $x>0$.

Maximum and minimum points, etc.: We have $\dfrac{dy}{dx} = \dfrac{1-\ln x}{x^2}$ and $\dfrac{d^2y}{dx^2} = \dfrac{2\ln x - 3}{x^3}$. Hence, the critical point is $(e, 1/e)$. At that point, $d^2y/dx^2 = -1/e^3 < 0$; so we have a relative maximum.

There is a point of inflection for $2\ln x = 3$, that is, at $(e^{3/2}, 3/2e^{3/2})$. The curve is concave downward for $0<x<e^{3/2}$ and concave upward for $x>e^{3/2}$.

Asymptotes: The y axis is a vertical asymptote, since $\dfrac{\ln x}{x} \to -\infty$ as $x \to 0^+$. By l'Hospital's rule, $\dfrac{\ln x}{x} \to 0$ as $x \to +\infty$. Hence, the positive x axis is a horizontal asymptote.

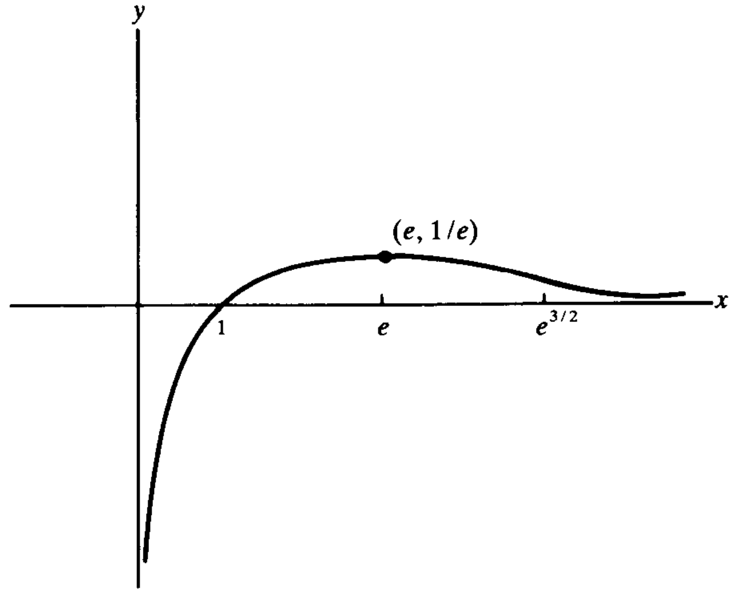


Fig. 29-6

Supplementary Problems

In Problems 7 to 38, discuss and sketch the curve.

7. $(x-2)(x-6)y = 2x^2$

8. $x(3-x^2)y = 1$

9. $(1-x^2)y = x^4$

10. $xy = (x^2-9)^2$

11. $2xy = (x^2-1)^3$

12. $x(x^2-4)y = x^2-6$

13. $y^2 = x(x^2-4)$

14. $y^2 = (x^2-1)(x^2-4)$

15. $xy^2 = x^2+3x+2$

16. $(x^2-2x-3)y^2 = 2x+3$

17. $x(x-1)y = x^2-4$

18. $(x+1)(x+4)^2y^2 = x(x^2-4)$

19. $y^2 = 4x^2(4-x^2)$

20. $y^2 = 5x^4+4x^5$

21. $y^3 = x^2(8-x^2)$

22. $y^3 = x^2(3-x)$

23. $(x^2-1)y^3 = x^2$

24. $(x-3)y^3 = x^4$

25. $(x-6)y^2 = x^2(x-4)$

26. $(x^2-16)y^2 = x^3(x-2)$

27. $(x^2+y^2)^2 = 8xy$

28. $(x^2+y^2)^3 = 4x^2y^2$

29. $y^4-4xy^2 = x^4$

30. $(x^2+y^2)^3 = 4xy(x^2-y^2)$

31. $y^2 = x(x-3)^2$

32. $y^2 = x(x-2)^3$

33. $3y^4 = x(x^2-9)^3$

34. $x^3y^3 = (x-3)^2$

35. $y = x\ln x$

36. $y = 1/x - \ln x$

37. $y = e^x/x$

38. $y = x^{2/3} - x^{5/3}$